X_Σ is separated:

Proposition: X_Σ is separated.

Proof: We start with the algebraic geometry lemma. $\{U_\sigma : \sigma \in \Sigma\}$ is an affine cover with $U_\tau = U_{\sigma_1} \cap U_{\sigma_2}$ if $\tau = \sigma_1 \cap \sigma_2$ by construction. We just need to see that

$$\mathbb{C}[S_{\sigma_1}] \otimes_{\mathbb{C}} \mathbb{C}[S_{\sigma_2}] \to \mathbb{C}[S_\tau]$$

is surjective. This holds if for every $m \in S_\tau$ there is an $m_1 \in S_{\sigma_1}$, $m_2 \in S_{\sigma_2}$ with $m = m_1 + m_2$, that is, if $S_\tau \subset S_{\sigma_1} + S_{\sigma_2}$. (Incidentally, it's clear that $S_{\sigma_1} + S_{\sigma_2} \subset S_\tau$.)

Now we use the polyhedral geometry lemmas. Let $u \in \mathrm{RelInt}((\sigma_1 - \sigma_2)^\vee)$, so $\tau = \sigma_1 \cap u^\perp = \sigma_2 \cap u^\perp$. Since $\{0\}$ is in both σ_1 and $-\sigma_2$, we have $\sigma_1 \subset (\sigma_1 - \sigma_2)$, $-\sigma_2 \subset (\sigma_1 - \sigma_2)$, and $(\sigma_1 - \sigma_2)^\vee \subset (\sigma_1^\vee \cap (-\sigma_2)^\vee)$. Then $u \in \sigma_1^\vee$ and $(-\sigma_2)^\vee$.

Taking u integral, we have $u \in S_{\sigma_1}$, $-u \in S_{\sigma_2}$, and $\tau = \sigma_1 \cap u^\perp$.

So $S_\tau = S_{\sigma_1} + \mathbb{Z}_{\geq 0} \cdot (-u) \subset S_{\sigma_1} + S_{\sigma_2}$. ∎

Survey of important results we don't have time to cover properly:

Orbit-Cone Correspondence:

Toric varieties are stratified by torus orbits/orbit closures in a beautiful way.

Proposition:
- There is a 1-1 correspondence: $\{\sigma \in \Sigma\} \longleftrightarrow \{T_N\text{-orbits } O(\sigma) \text{ in } X_\Sigma\}$.
- $O(\sigma) \cong \mathrm{Spec}(\mathbb{C}[\sigma^\perp \cap M])$ ← If σ is a cone of dimension r, $O(\sigma)$ is a torus of codim r.
- $U_\sigma = \bigcup_{\tau \text{ face of } \sigma} O(\tau)$
- τ is a face of σ $\Leftrightarrow$ $O(\sigma)$ is contained in the orbit closure $V(\tau) := \overline{O(\tau)}$.
- $V(\tau) = \bigcup_{\tau \text{ face of } \sigma} O(\sigma)$ is a toric variety with open torus $O(\tau)$.

Question: Can you describe the fan of $V(\tau)$?

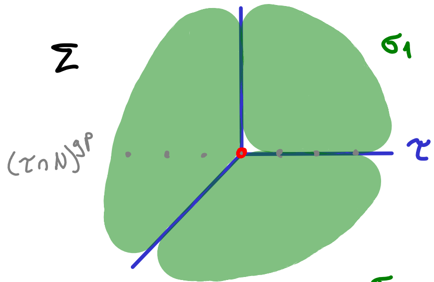
Answer: The open torus is $\mathrm{Spec}(\mathbb{C}[\tau^\perp \cap M])$, so the character lattice is $\tau^\perp \cap M$ and cocharacter lattice is its dual: $N/(\tau^\perp \cap M)^\perp = N/(\tau \cap N)^{\mathrm{gp}} =: N(\tau)$.

So we're looking for a fan in $N(\tau)$.

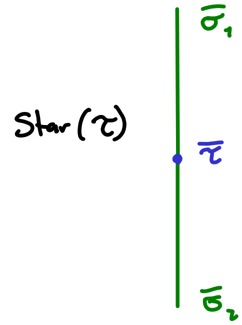
The cones: for $\sigma \in \Sigma$ with τ a face of σ, consider $\bar{\sigma}$ the image of σ in $N(\tau)$.

These $\bar{\sigma}$ form a fan called $\mathrm{Star}(\tau)$.

Example:



$X_\Sigma = \mathbb{P}^2$



$X_{\mathrm{Star}(\tau)} = V(\tau) = \mathbb{P}^1$

Morphisms of toric varieties

Question: How should a morphism of toric varieties be defined?

Answer: There are 2 ingredients that should play a role:

- Toric varieties are schemes — should be a scheme morphism.
- They come with an open dense torus acting on the whole scheme — morphism should respect these inclusions and actions.

Def: Let $T_{N_1} \subset X_{\Sigma_1}$ and $T_{N_2} \subset X_{\Sigma_2}$ be toric varieties. Then a **morphism of toric varieties**

$$(f, f^\#) : (X_{\Sigma_1}, \mathcal{O}_{X_{\Sigma_1}}) \to (X_{\Sigma_2}, \mathcal{O}_{X_{\Sigma_2}})$$

is a morphism of schemes such that

- $f(T_{N_1}) \subset T_{N_2}$ and
- the restriction of $(f, f^\#)$ to $(T_{N_1}, \mathcal{O}_{X_{\Sigma_1}}|_{T_{N_1}}) \to (T_{N_2}, \mathcal{O}_{X_{\Sigma_2}}|_{T_{N_2}})$ is a homomorphism of affine algebraic group schemes.

Observe: We have a commutative diagram in this case:

$$\begin{array}{ccc} T_{N_1} \times_{\mathrm{Spec}(\mathbb{C})} X_{\Sigma_1} & \xrightarrow{\mu_1} & X_{\Sigma_1} \\ {\scriptstyle f\otimes f}\downarrow & & \downarrow{\scriptstyle f} \\ T_{N_2} \times_{\mathrm{Spec}(\mathbb{C})} X_{\Sigma_2} & \xrightarrow{\mu_2} & X_{\Sigma_2} \end{array}$$

" $f : X_{\Sigma_1} \to X_{\Sigma_2}$ is an **equivariant morphism** with respect to the T_{N_1} and T_{N_2} actions"

Question: Is there some convex polyhedral geometry description of a morphism of toric varieties $f : X_{\Sigma_1} \to X_{\Sigma_2}$?

- f restricts to a homomorphism of affine algebraic group schemes
- $\mathrm{Hom}_{\text{Alg gp sch}}(T_{N_1}, T_{N_2}) = \mathrm{Hom}_{\mathbb{Z}}(N_1, N_2)$ ← Same arguments used to describe character and cocharacter lattices of T_N.

$\Rightarrow$ Should involve a $\mathbb{Z}$-linear map from N_1 to N_2.

Consider $U_{\sigma_1} \to U_{\sigma_2}$. This corresponds to a semigroup homomorphism $S_{\sigma_2} \xrightarrow{\psi} S_{\sigma_1}$, so a linear map $M_{2,\mathbb{R}} \xrightarrow{\psi} M_{1,\mathbb{R}}$ sending $\sigma_2^\vee$ to $\sigma_1^\vee$. Take $n \in \sigma_1$. Then for $m \in \sigma_2^\vee$,

$\langle \psi^*(n), m \rangle_{N_2 \times M_2} = \langle n, \psi(m) \rangle_{N_1 \times M_1} \geq 0$ and $\psi^*(\sigma_1) \subset \sigma_2$.

Candidate polyhedral geometry description:

$$\left\{ \begin{array}{c} \text{Morphisms of toric varieties} \\ f: X_{\Sigma_1} \to X_{\Sigma_2} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \mathbb{Z}\text{-linear maps } \varphi: N_1 \to N_2 \text{ where for every} \\ \sigma_1 \in \Sigma_1 \text{ there is a } \sigma_2 \in \Sigma_2 \text{ with } \varphi(\sigma_1) \subset \sigma_2 \end{array} \right\}$$

← obvious extension to $N_{1,\mathbb{R}} \to N_{2,\mathbb{R}}$

Def: Such a $\varphi: N_1 \to N_1$ is said to be **compatible with Σ_1 and Σ_2**.

Proposition: If $\varphi: N_1 \to N_2$ is a $\mathbb{Z}$-linear compatible with the fans Σ_1 and Σ_2, it defines a morphism of toric varieties $f: X_{\Sigma_1} \to X_{\Sigma_2}$.

Proof: For each affine open toric subvariety $U_{\sigma_1} \subset X_{\Sigma_1}$, we have some σ_2 with $\varphi(\sigma_1) \subset \sigma_2$. Then for $n \in \sigma_1$ and $m \in S_{\sigma_2}$ we have $\langle n, \varphi^*(m) \rangle = \langle \varphi(n), m \rangle \geq 0$, so $\varphi^*: M_2 \to M_1$ restricts to a semigroup homomorphism $S_{\sigma_2} \to S_{\sigma_1}$. It induces a morphism of affine toric varieties $f_{\sigma_1}: U_{\sigma_1} \to U_{\sigma_2}$. These f_{σ_1}, $\sigma_1 \in \Sigma_1$, glue: if $\tau \subset \sigma_1, \sigma_1'$ then $f_{\sigma_1}|_{U_\tau} = f_\tau = f_{\sigma_1'}|_{U_\tau}$. ∎

Proposition: If $f: X_{\Sigma_1} \to X_{\Sigma_2}$ is a morphism of toric varieties, it defines a $\mathbb{Z}$-linear map $\varphi: N_1 \to N_2$ compatible with Σ_1 and Σ_2.

Proof: As we have seen, restriction of f to a homomorphism of affine algebraic group schemes $T_{N_1} \to T_{N_2}$ gives a $\mathbb{Z}$-linear map $\varphi: N_1 \to N_2$.

For compatibility: use equivariance and the orbit-cone correspondence. Want to see that these together imply $f(U_{\sigma_1})$ is contained in some U_{σ_2}. Hence we have a restriction $U_{\sigma_1} \to U_{\sigma_2}$, but these affine toric morphisms are induced by semigroup homomorphisms $S_{\sigma_2} \to S_{\sigma_1}$, so $\varphi(\sigma_1) \subset \sigma_2$.

So, Consider $O(\sigma_1) \subset X_{\Sigma_1}$. By equivariance, $f(O(\sigma_1))$ must be contained in some torus orbit $O(\sigma_2) \subset X_{\Sigma_2}$. If τ_1 is a face of σ_1, we also have $f(O(\tau_1))$ contained in some $O(\tau_2) \subset X_{\Sigma_2}$.

But $O(\sigma_1) \subset V(\tau_1)$ and $f(V(\tau_1)) \subset V(\tau_2)$, so $f(O(\sigma_1)) \subset V(\tau_2) = \bigcup_{\tau_2 \text{ face of } \sigma_2'} O(\sigma_2')$.

Then τ_2 must be a face of σ_2.

So $f(U_{\sigma_1}) = f\left(\bigcup_{\tau_1 \text{ face of } \sigma_1} O(\tau_1)\right) \subset \bigcup_{\tau_2 \text{ face of } \sigma_2} O(\tau_2) = U_{\sigma_2}$. ∎

Projective Toric Varieties

The Proj construction

Idea – While Spec takes a ring R as input and spits out an affine scheme X whose coordinate ring is R, Proj takes a graded ring $S = \bigoplus_{j=0}^{\infty} S_j$ as input and spits out a projective variety with a line bundle whose section ring is S.

Def: Let $S = \bigoplus_{j \geq 0} S_j$ be a graded ring. The **irrelevant ideal** is $S_+ := \bigoplus_{j > 0} S_j$.

The set **Proj(S)** consists of the homogeneous prime ideals of S not containing S_+.

It is equipped with a topology by defining closed sets to be of the form

$$V(I) = \{ p \in \mathrm{Proj}(S) : I \subset p \}.$$

Further details and scheme structure in Hartshorne Section II.2.

The result is a projective variety over the ring S_0.

Example: Let $S = \mathbb{C}[x,y]$, graded by total degree. Then $S_+ = \{ f \in S : f(0,0) = 0 \}$.

Homogeneous maximal ideals not containing S_+ are of the form

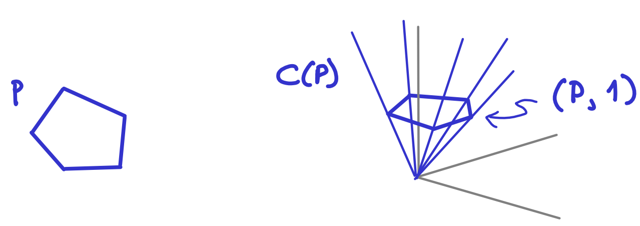
$$\langle bx - ay : (a,b) \in \mathbb{C}^2 \setminus \{0\} \rangle.$$

Note that $\langle \lambda b x - \lambda a y \rangle = \langle bx - ay \rangle$ for $\lambda \in \mathbb{C}^*$. $\mathrm{Proj}(S) = \mathbb{P}^1$.

S is the section ring of $\mathcal{O}_{\mathbb{P}^1}(1)$. That is, $S = \bigoplus_{j \geq 0} \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)^{\otimes j})$.

The Toric Case

Let P be a full dimensional rational convex polytope in $M_{\mathbb{R}}$, and let $C(P)$ be the cone over P in $M_{\mathbb{R}} \oplus \mathbb{R}$:



Observe that $C(P) \cap (M \oplus \mathbb{Z})$ is a $\mathbb{Z}$-graded semigroup, and $S_P := \mathbb{C}[C(P) \cap (M \oplus \mathbb{Z})]$ is a $\mathbb{Z}$-graded $\mathbb{C}$-algebra.

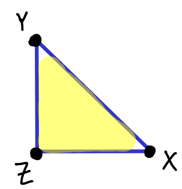
More generally, if P is a full dimensional rational convex polyhedron (not necessarily bounded), $\overline{C(P)} \cap (M \oplus \mathbb{Z})$ is a $\mathbb{Z}$-graded semigroup, and $S_P := \mathbb{C}[\overline{C(P)} \cap (M \oplus \mathbb{Z})]$ is a $\mathbb{Z}$-graded $S_{P,0}$ algebra, where $S_{P,j}$ is the degree j homogeneous subspace.

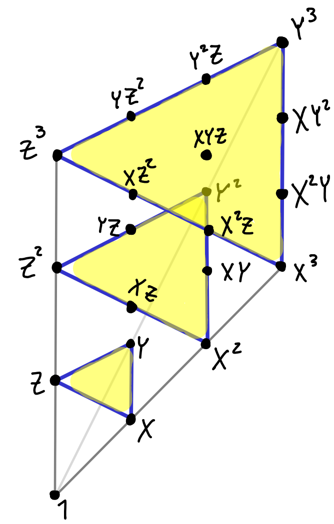
subring generated by degree 0 elements

Question: If $X_P := \mathrm{Proj}(S_P)$ is a toric variety, with S_P the section ring of a line bundle on X_P, how can we describe the defining torus?

Answer: Let $P' = M_{\mathbb{R}}$. Then $X_{P'}$ is naturally identified with $\mathrm{Spec}(\mathbb{C}[M])$.

Example:

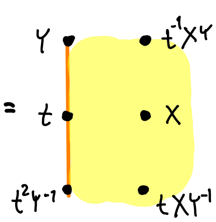
$P =$ 

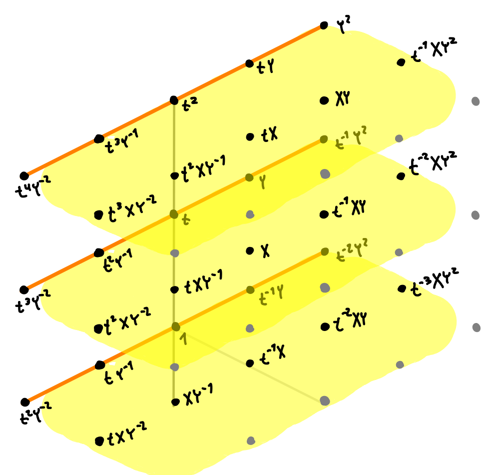
$\overline{C(P)} = C(P) =$ 

Question: $X_P = ?$ Line bundle?

Answer: $\mathbb{P}^2$. $\mathcal{O}_{\mathbb{P}^2}(1)$.

Example:

$P =$ 

$\overline{C(P)} =$ 

Question: $X_P = ?$

Answer: $\mathbb{P}^1_{\mathbb{C}[Y, Y^{-1}]} \cong \mathbb{C} \times \mathbb{C}^*$

As schemes over $\mathbb{C}$

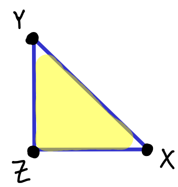
Relating Polyhedra and Fans

Let $V \subset \mathbb{P}^n$ be a projective variety. The atlas $\{U_I := \{x_i \neq 0 : i \in I\} : I \subset \{0, \dots, n\}\}$ of affine open subvarieties of $\mathbb{P}^n$ induces an atlas $\{V_I := V \cap U_I : I \subset \{0, \dots, n\}\}$ of affine open subvarieties of V.

This means P is associated to a "very ample" line bundle. Not essential, but simplifies the picture.

Suppose the sections associated to $P \cap M$ give an embedding of X_P into $\mathbb{P}(S_{P,1})$. Then at each point $p \in X_P$, some section s is non-vanishing and the non-vanishing locus of this section is an affine open subvariety.

Example:

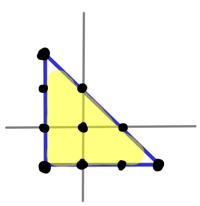
$P =$ 

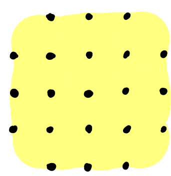
$$\overset{z \neq 0}{\rightsquigarrow} \quad \mathbb{C}^2_{\frac{x}{z}, \frac{y}{z}} \subset \mathbb{P}^2_{[x:y:z]}$$

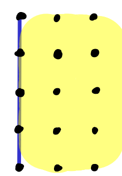
Division by z^m is subtraction of exponent vectors. Natural description of affine patches:

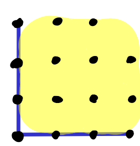
For $m \in P \cap M$, let $C_m = \operatorname{Span}_{\mathbb{R}_{\geq 0}}(P - m)$ and consider $\operatorname{Spec}(\mathbb{C}[C_m \cap M])$.

Example:

$P =$ 

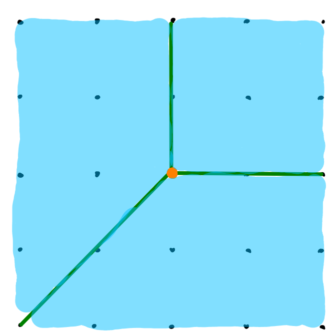
$C_{(0,0)} =$ 

$C_{(-1,0)} = C_{(-1,1)} =$  ...

$C_{(-1,-1)} =$  ...

Question: Do you recognize these cones?

Answer: They are the dual cones to the fan from Problem Set 2, the fan for $\mathbb{P}^2$.



This is a general phenomenon. Let P be a full dimensional rational convex polyhedron.

Let $\mathcal{P}$ be the set $\{P\} \cup \{\text{Faces of } P\}$, so $P = \coprod_{F \in \mathcal{P}} \mathrm{RelInt}(F)$.

Proposition:

- If $F \in \mathcal{P}$ and $x, y \in \mathrm{RelInt}(F)$, then $C_x = C_y$, and $\mathrm{Spec}(\mathbb{C}[C_x \cap M]) = \mathrm{Spec}(\mathbb{C}[C_y \cap M])$.

  Denote this cone C_F.

- The cones $\{C_F^\vee : F \in \mathcal{P}\}$ are strongly convex and form a fan Σ_P in N.

  Σ_P is called the **normal fan of P.**

- We have $X_P = X_{\Sigma_P}$.

See Cox-Little-Schenck Section 7.1.