X_Σ is separated:

Some background on scheme morphisms:

Def: Let $f: X \to Y$ be a continuous map and $\mathcal{F}$ a sheaf on X. Then the **direct image sheaf** $f_*\mathcal{F}$ on Y is defined by setting $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ for any open subset $U \subset Y$.

Def: Let R and S be local rings. (Recall that this means each has a unique maximal ideal.)
A ring homomorphism $\phi: R \to S$ is a **local homomorphism** if $\phi^{-1}(m_S) = m_R$.

Def: Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be schemes. Then a **morphism**

$$(f, f^\#): (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

is continuous map $f: X \to Y$ together with $f^\#: \mathcal{O}_Y \to f_*\mathcal{O}_X$ a map of sheaves on Y such that for each $p \in X$, the induced map of local rings $f^\#_p: \mathcal{O}_{Y,f(p)} \longrightarrow \mathcal{O}_{X,p}$ is a local homomorphism. ← This essentially says that if local function g vanishes at $f(p)$, then f^*g vanishes at p.

Def: A morphism of schemes $(i, i^\#): (Z, \mathcal{O}_Z) \longrightarrow (X, \mathcal{O}_X)$ is a **closed immersion** if i induces a homeomorphism of Z with its image (as topological spaces) and $i^\#: \mathcal{O}_X \to i_*\mathcal{O}_Z$ is a surjection. ← This essentially says local functions on Z – something in $i_*\mathcal{O}_{Z,p}$ here – extend to local functions on X – something in $\mathcal{O}_{X,i(p)}$.

Gluing is **always** the source of non-separatedness – affine schemes are necessarilly separated. (See Hartshorne Prop II.4.1, or Shafarevich Prop 5.3) Moreover:

An Algebraic Geometry Lemma:

Lemma: Let X be a scheme over $\mathbb{C}$ with $\{U_\alpha := \mathrm{Spec}(R_\alpha)\}$ an affine open cover such that

- $U_{\alpha\beta} := U_\alpha \cap U_\beta$ is also affine, say $U_{\alpha\beta} = \mathrm{Spec}(R_{\alpha\beta})$
- $R_\alpha \otimes_{\mathbb{C}} R_\beta \longrightarrow R_{\alpha\beta}$ is surjective.
  $r \otimes s \longmapsto rs$

Then X is separated.

Proof: (See Shafarevich Prop 5.4)

Some convex polyhedral geometry lemmas:

Observation: Faces τ of σ have the form $\tau = \sigma \cap u^\perp$ for some $u \in \sigma^\vee \cap M$.

Lemma: If $\tau = \sigma \cap u^\perp$ for some $u \in \sigma^\vee$, then $\tau^\vee = \sigma^\vee + \mathbb{R}_{\geq 0} \cdot (-u)$.

Proof: $\tau \subset (\sigma^\vee + \mathbb{R}_{\geq 0} \cdot (-u))^\vee$

Suppose $n \in \tau$. Then $\langle n, m \rangle \geq 0$ for all $m \in \sigma^\vee$ and $\langle n, u \rangle = 0$. So $\langle n, m + \alpha u \rangle \geq 0$ for all $m + \alpha u \in \sigma^\vee + \mathbb{R} \cdot u = \sigma^\vee + \mathbb{R}_{\geq 0} \cdot (-u)$, and $n \in (\sigma^\vee + \mathbb{R}_{\geq 0} \cdot (-u))^\vee$.

$(\sigma^\vee + \mathbb{R}_{\geq 0} \cdot (-u))^\vee \subset \tau$

Suppose $n \in (\sigma^\vee + \mathbb{R}_{\geq 0} \cdot (-u))^\vee$. Then $\langle n, m - \alpha u \rangle \geq 0$ for all $m \in \sigma^\vee$, $\alpha \in \mathbb{R}_{\geq 0}$. With $\alpha = 0$, this implies $\langle n, m \rangle \geq 0$ for all $m \in \sigma^\vee$, so $n \in (\sigma^\vee)^\vee = \sigma$. If we fix $m \in \sigma^\vee$ and take α arbitrarily large, we see that $\langle n, -u \rangle \geq 0$, so $n \in (-u)^\vee$.

Then $n \in \sigma^\vee \cap (-u)^\vee = \sigma^\vee \cap u^\perp$ since $u \in \sigma$, so $\sigma^\vee \subset u^\vee$. $n \in \tau$. ∎

Lemma: If $\tau = \sigma \cap u^\perp$ for some $u \in S_\sigma$, then $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0} \cdot (-u)$.

Proof: First, $S_\tau = \tau^\vee \cap M = (\sigma^\vee + \mathbb{R}_{\geq 0} \cdot (-u)) \cap M$.

Clearly, $S_\sigma + \mathbb{Z}_{\geq 0} \cdot (-u) = (\sigma^\vee \cap M) + \mathbb{Z}_{\geq 0} \cdot (-u) \subset (\sigma^\vee + \mathbb{R}_{\geq 0} \cdot (-u)) \cap M = S_\tau$.

Next, take $v \in S_\tau$. Then $v = m - \alpha u$ for some $m \in \sigma^\vee$, $\alpha \in \mathbb{R}_{\geq 0}$.

Then $v + \beta u \in \sigma^\vee$ whenever $\beta \geq \alpha$. If furthermore $\beta \in \mathbb{Z}$, then $v + \beta u =: m' \in S_\sigma$.

So $v = m' - \beta u$ is in $S_\sigma + \mathbb{Z}_{\geq 0} \cdot (-u)$. ∎

Lemma: Let $\tau = \sigma_1 \cap \sigma_2$. If u is in the relative interior of $(\sigma_1 - \sigma_2)^\vee$, then $\tau = \sigma_1 \cap u^\perp = \sigma_2 \cap u^\perp$.

Minkowski sum of σ_1 and $-\sigma_2$.

Proof: Note that for a convex cone $C \subset M_\mathbb{R}$, the relative interior of C is characterized by

$$\mathrm{RelInt}(C) = \{ m \in C : \langle n, m \rangle > 0 \text{ for all } n \in C^\vee \setminus C^\perp \}.$$

So, $m \in \mathrm{RelInt}(C)$ implies $C^\vee \cap m^\perp = C^\perp = C^\vee \cap (-C)^\vee$. Taking $C = (\sigma_1 - \sigma_2)^\vee$ and $m = u$, we have $(\sigma_1 - \sigma_2) \cap u^\perp = (\sigma_1 - \sigma_2) \cap (\sigma_2 - \sigma_1)$.

Claim: $(\sigma_1 - \sigma_2) \cap (\sigma_2 - \sigma_1) = \tau - \tau$.

- If $n, n' \in \tau = \sigma_1 \cap \sigma_2$, then $n - n' \in (\sigma_1 - \sigma_2) \cap (\sigma_2 - \sigma_1)$.
- If $n_1 - n_2 = n_2' - n_1' \in (\sigma_1 - \sigma_2) \cap (\sigma_2 - \sigma_1)$, then $n_1 + n_1' = n_2 + n_2' \in \sigma_1 \cap \sigma_2 = \tau$.

  But $n_i, n_i' \in \sigma_i$ and $n_i + n_i' \in \tau$ implies $n_i, n_i' \in \tau$ (since τ is a face of σ_i).

  So $n_1 - n_2 \in \tau - \tau$.

We have that $(\sigma_1 - \sigma_2) \cap u^\perp = \tau - \tau$. Next,

$\sigma_1 \cap u^\perp = \sigma_1 \cap (\sigma_1 - \sigma_2) \cap u^\perp = \sigma_1 \cap (\tau - \tau) = \tau$ and

$-\sigma_2 \cap u^\perp = -\sigma_2 \cap (\sigma_1 - \sigma_2) \cap u^\perp = -\sigma_2 \cap (\tau - \tau) = -\tau$, so $\sigma_2 \cap u^\perp = \tau$. ∎