Last week in Toric Varieties ...

A distinguished point:

Proposition: Let $\sigma \subset N_{\mathbb{R}}$ be full dimensional. Then the $\mathbb{C}$-algebra homomorphism defined by

$$\mathbb{C}[S_\sigma] \xrightarrow{\varphi} \mathbb{C}$$
$$z^m \longmapsto \begin{cases} 1 & \text{if } m \in \sigma^\perp \\ 0 & \text{otherwise} \end{cases}$$

determines a closed point $x_\sigma \in U_\sigma$ which is fixed by the T_N-action.

Proof: By definition, x_σ is the prime ideal $\varphi^{-1}(0)$ in $\mathbb{C}[S_\sigma]$. Since σ is full dimensional, $\sigma^\vee$ is strongly convex and $\sigma^\perp = \{0\} \subset M_{\mathbb{R}}$. That is, $\varphi(z^0) = 1$ while $\varphi(z^m) = 0$ for $m \neq 0$, and x_σ is the **maximal** ideal of non-invertible elements in $\mathbb{C}[S_\sigma]$ — it is a closed point.

Next, viewed classically as a point in U_σ, x_σ is the common vanishing locus of $\{z^m : m \in S_\sigma \setminus \{0\}\}$. For $t \in T_N$, $m \in S_\sigma \setminus \{0\}$, we have

$$z^m(t \cdot x_\sigma) = z^m(t) z^m(x_\sigma) = z^m(t) \cdot 0 = 0,$$

so $t \cdot x_\sigma = x_\sigma$. ∎

Smooth affine toric varieties:

Def: Let $X = \mathrm{Spec}(R)$ be an affine scheme and $\mathfrak{m} \in X$ a closed point. The **cotangent space of X at $\mathfrak{m}$** is $\mathfrak{m}/\mathfrak{m}^2$.

The idea here is that the ideal $\mathfrak{m}$ consists of those local functions $f \in \mathcal{O}_{X,\mathfrak{m}}$ which vanish at the point $\mathfrak{m} \in X$. Each of these fuctions defines a map on the tangent space, so an element of the cotangent space. Two such functions define the same map on the tangent space if they agree to first order, so $\mathfrak{m}/\mathfrak{m}^2$. However, this definition applies more generally — e.g. when the tangent space is not defined.

Def: Let $X = \mathrm{Spec}(R)$ be an affine scheme. X is **smooth** if for every closed point $\mathfrak{m} \in X$, we have $\dim(X) = \dim(\mathfrak{m}/\mathfrak{m}^2)$.

Example: $T_N = \mathrm{Spec}(\mathbb{C}[M])$, $M \cong \mathbb{Z}^2$. Take $\mathfrak{m} = (x-a, y-b)$ for some $a, b \in \mathbb{C}^*$. Then $\mathfrak{m}^2 = ((x-a)^2, (x-a)(y-b), (y-b)^2)$, and $\mathfrak{m}/\mathfrak{m}^2 = \{ \lambda_1 \overline{(x-a)} + \lambda_2 \overline{(y-b)} : \lambda_i \in \mathbb{C}\}$.

$\dim(T_N) = 2 = \dim(\mathfrak{m}/\mathfrak{m}^2)$.

Non-example: (Cuspidal cubic plane curve) $X = \mathrm{Spec}(\mathbb{C}[x,y]/\langle x^3 - y^2\rangle)$.

Take $\mathfrak{m} = (x,y)$. Then $\mathfrak{m}^2 = (x^2, xy, y^2)$, and $\mathfrak{m}/\mathfrak{m}^2 = \{\lambda_1 \bar{x} + \lambda_2 \bar{y} : \lambda_i \in \mathbb{C}\}$.

$\dim(X) = 1$ but $\dim(\mathfrak{m}/\mathfrak{m}^2) = 2$.

Proposition: U_σ is smooth if and only if the set of primitive ray generators $\{n_i\}$ of σ is a subset of some $\mathbb{Z}$-basis of N.

Proof: First suppose σ is full dimensional. Then U_σ has a unique T_N-fixed point x_σ. Denote the associated maximal ideal by $\mathfrak{m}$. Recall $\mathfrak{m} = (\chi^m : m \in S_\sigma \setminus \{0\})$. Then $\mathfrak{m}^2 = (\chi^{m+m'} : m, m' \in S_\sigma \setminus \{0\})$, and

$$\mathfrak{m}/\mathfrak{m}^2 = \{\textstyle\sum_i \lambda_i \overline{\chi^{m_i}} : \lambda_i \in \mathbb{C},\ m_i \in S_\sigma \setminus \{0\},\ m_i \neq m + m' \text{ for some } m, m' \in S_\sigma \setminus \{0\}\}.$$

The primitive ray generators of $\sigma^\vee$ all define basis elements of $\mathfrak{m}/\mathfrak{m}^2$. But $\sigma^\vee$ is a full dimensional strongly convex cone, so smoothness implies there are exactly $\dim(U_\sigma)$ of these and they define **all** basis elements of $\mathfrak{m}/\mathfrak{m}^2$. That is, these ray generators of $\sigma^\vee$ must generate S_σ as a semigroup, and in turn M as a group. They form a $\mathbb{Z}$-basis for M, so $\{n_i\}$ — the dual basis — is a basis for N.

Similarly, if $\{n_i\}$ is a $\mathbb{Z}$-basis for N, then

$$\mathfrak{m}/\mathfrak{m}^2 = \mathrm{Span}_{\mathbb{C}}\{\overline{\chi^{m_i}} : m_i \text{ in the dual basis for } M\}.$$

Hence $\dim(U_\sigma) = \dim(\mathfrak{m}/\mathfrak{m}^2)$, and x_σ is a smooth point.

As you (hopefully) concluded in the problem set, every other closed point $y \in U_\sigma$ is contained in some U_τ for τ a face of σ. So, we move on to the case σ ***not*** full dimensional.

Define $N_\sigma := \sigma \cap N + (-\sigma \cap N)$. This is a saturated sublattice of N, so we can find a splitting $N = N_\sigma \oplus N''$ and write σ as $\sigma' \oplus \{0\}$, where σ' is full dimensional in N_σ. Decomposing M similarly as $M = M' \oplus M''$, we have $S_\sigma = S_{\sigma'} \oplus M''$, and

$$U_\sigma \cong U_{\sigma'} \times T_{N''}.$$

But $U_{\sigma'} \times T_{N''}$ is smooth if and only if $U_{\sigma'}$ is. We are back to the case of a full dimensional cone. Using the splitting, we extend a basis for N_σ to a basis for N. ∎

Fans – building toric varieties with an atlas:

Def: A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ such that for each point $p \in X$ there is an open set $U \subset X$ containing p with $(U, \mathcal{O}_X|_U)$ an affine scheme.

Def: A **fan** Σ in N is a collection of strongly convex cones $\sigma \subset N_{\mathbb{R}}$ such that:

- if $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$.
- if $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.
- there are finitely many cones $\sigma \in \Sigma$.

optional. With this condition, we'll study **finite type** toric varieties. Without it – **locally finite type**.

Example: Problem 1.

Def: The toric variety X_Σ is the scheme with affine open toric subvarieties U_σ for $\sigma \in \Sigma$, and with U_{σ_1} and U_{σ_2} glued by the inclusion of U_τ into each if $\tau = \sigma_1 \cap \sigma_2$.

Example: Problem 1.

X_Σ is "separated":

Atlases are good for making definitions, but in practice can be a terrible way to construct a space – end result may be very ugly.

Def: Let X be a scheme over $\mathrm{Spec}(\mathbb{C})$. The **diagonal morphism** is the unique scheme morphism $\Delta : X \longrightarrow X \times_{\mathrm{Spec}(\mathbb{C})} X$ with $pr_1 \circ \Delta = id_X = pr_2 \circ \Delta$.

$$\begin{tikzcd}
X \arrow[dr, dashed, "\Delta"] \arrow[drr, bend left, "id_X"] \arrow[ddr, bend right, "id_X"'] & & \\
& X \times_{\mathrm{Spec}(\mathbb{C})} X \arrow[r, "pr_2"] \arrow[d, "pr_1"] & X \arrow[d] \\
& X \arrow[r] & \mathrm{Spec}(\mathbb{C})
\end{tikzcd}$$

X is **separated** if Δ is a closed immersion. ⟵ To be defined soon.

Intuition from topology: "Separated" is the algebraic geometry version of "Hausdorff".

Remember, X is Hausdorff if for every pair of points $p_1, p_2 \in X$ there is a pair of open sets $U_1, U_2 \subset X$ with $p_i \in U_i$ and $U_1 \cap U_2 = \emptyset$.

The standard non-example is the line with 2 origins: Let $U_i = \mathbb{R}$, and $V_i = \mathbb{R} \setminus \{0\}$. construct X by gluing U_1 and U_2 along V_1 and V_2 via the identity map on $\mathbb{R} \setminus \{0\}$.

Then any open set containing 1 origin must contain the other. (We can replace $\mathbb{R}$ by $\mathbb{C}$ here too.)

Observation: X is Hausdorff if and only if $\Delta(X)$ is closed in $X \times X$.

If X is not Hausdorff, then there is a pair of points $p_1, p_2 \in X$ that cannot be separated by open sets. Then $(p_1, p_2) \in X \times X$ is contained in $\overline{\Delta(X)}$ but not in $\Delta(X)$.

Next week in Toric Varieties:

- Why is X_Σ separated?
- Toric Morphisms