Why toric varieties?

- Beautiful theory where algebraic geometry (hard) and convex polyhedral geometry (relatively simple) coincide

  The theory of normal, finite type toric varieties is pretty well-developed now in comparison to adjacent fields. There are still interesting open questions, but don't expect to find much low-hanging fruit left here. → You get a less studied theory if you drop these adjectives.

- Gateway to other areas of algebraic geometry:
  - Compactifications of moduli spaces

    Enumerative geometry is simpler with compact than non-compact spaces:

    - How many points are in the intersection of a pair of distinct lines in the
      i) affine plane?
      ii) projective plane?

    So compactifications of moduli spaces are natural to consider, but not all compactifications are created equal. A particularly nice class are "toroidal compactifications", which locally look like toric varieties. Toric machinery can be applied here.
    (As a historical note, the theory of toric varieties was developed for this purpose.)

  - Toric degenerations

    Typical set-up: Want to find some property of a polarized projective variety. Hard to determine directly, but easy in the toric case. If this property is preserved in flat families, can try to build a flat family where some fiber is the variety of interest and another fiber is toric. Hilbert polynomials and degree may be computed this way.

  - Mirror Symmetry

    Toric geometry plays a central role in multiple mirror symmetry constructions. In Batyrev (-Borisov) constructions, mirror families of Calabi-Yau varieties are built within dual ambient toric varieties.

There are related constructions involving toric degenerations to one member of a pair of Batyrev dual toric varieties on one side, and blow-ups of the boundary of the other member on the mirror side. This has close ties to the FanoSearch program based at Imperial.

- Log Calabi-Yau Varieties / Cluster Varieties

Toric varieties are the "baby case" of **partial minimal models for log Calabi-Yau varieties**. Cluster varieties could be considered the "adolescent case" of log Calabi-Yau varieties, but they are still a hot research topic with connections to many different areas of math. We can try to generalize the algebraic geometry — convex polyhedral geometry dictionary to these broader (non-toric) settings. This involves a notion of convexity in piecewise linear spaces (specifically, real tropical space) rather than real vector spaces. It would provide a great way to understand the geometry of these more general partial minimal models, and if realized it would be another beautiful theory analogous to toric geometry.
(Key area of research for me and my collaborators.)

Affine Schemes (mostly following Hartshorne)

The first task is to formalize the algebraic properties of functions on a topological space.

Bare minimum that should be required:

Def: A **presheaf of rings** $\mathcal{F}$ on a topological space X consists of the following data:

- a ring $\mathcal{F}(U)$ associated to each open subset $U \subseteq X$
- "restriction maps" a ring homomorphism $\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ for each inclusion $V \subseteq U$ of subsets of X

subject to the following conditions:

- ρ_{UU} is the identity map $\mathcal{F}(U) \to \mathcal{F}(U)$
- Restrictions compose: given $W \subseteq V \subseteq U$, $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$

Typically we will ask for a bit more structure:

Def: A **sheaf of rings** $\mathcal{F}$ is a presheaf of rings satisfying the following additional conditions:

- Sections determined by their restrictions: given an open subset $U \subseteq X$, open cover $\{V_i\}$ of U, and **sections** $s, t \in \mathcal{F}(U)$ with $\rho_{UV_i}(s) = \rho_{UV_i}(t)$ for all i, we have $s = t$.
- Local sections glue together: given an open subset $U \subseteq X$, open cover $\{V_i\}$ of U, and sections $s_i \in \mathcal{F}(V_i)$ for all i such that for each i and j, $\rho_{V_i, V_i \cap V_j}(s_i) = \rho_{V_j, V_i \cap V_j}(s_j)$, there is a section $s \in \mathcal{F}(U)$ with $\rho_{UV_i}(s) = s_i$ for all i.

At first glance it may seem impossible for these two conditions to fail. Here are two non-examples:

Non-example: Let X be the two point set $\{x, y\}$ with the discrete topology, and let $\mathcal{F}$ be the presheaf with $\mathcal{F}(X) = \mathbb{C}$, $\mathcal{F}(U) = 0$ for $U \subsetneq X$, and all non-identity restriction maps the 0 map. Then $\{\{x\}, \{y\}\}$ is an open cover of X, but $\rho_{X,\{x\}}(s) = 0 = \rho_{X,\{y\}}(s)$ for all $s \in \mathcal{F}(X)$ and not all such s are equal.

Non-example: Let $X = \mathbb{R}$ with the usual topology, and let $\mathcal{F}$ be the presheaf $\mathcal{F}(U) =$ bounded $\mathbb{R}$-valued functions on U, with the usual restriction maps. Let $\{V_i\}$ be a cover of bounded open intervals. Then, e.g., the local sections s_i sending $r \in V_i$ to $r \in \mathbb{R}$ are all perfectly well-defined elements of $\mathcal{F}(V_i)$, but there is no $s: \mathbb{R} \to \mathbb{R}$, $s(r) = r$ is not bounded.

Exercise: Pick your favorite complex variety. Carefully define its sheaf of holomorphic functions and verify that it is indeed a sheaf.

Def: Let p be a point in X and $\mathcal{F}$ a sheaf of rings on X. The **stalk** $\mathcal{F}_p$ of $\mathcal{F}$ at p is the direct limit of the rings $\mathcal{F}(U)$ for all open sets U containing p.
(Think functions on an arbitrarily small neighborhood of p.)

Def: A commutative ring with 1 is a **local ring** if it has a unique maximal ideal.
(The name comes from the fact that the stalks appearing in algebraic geometry have this form.)

Def: A **locally ringed space** is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X such that each stalk $\mathcal{O}_{X,p}$ is a local ring.

Def: Let R be a commutative ring with 1. Then **Spec(R)** is the topological space whose underlying set is the set of prime ideals of R and whose closed sets are described as follows:

for I an ideal of R, define the closed set $V(I) \subset \operatorname{Spec}(R)$ to be the set of prime ideals containing I.

Exercise: Show that this indeed defines a topology on $\operatorname{Spec}(R)$.

(Solution in Hartshorne Chapter II, section 2)

Def: The **structure sheaf $\mathcal{O}$ of Spec(R)** is defined as follows. For each open set $U \subset \operatorname{Spec}(R)$ define $\mathcal{O}(U)$ to be the set of functions $s: U \to \bigsqcup_{p \in U} R_p$ such that $s(p) \in R_p$ and there is some fraction of elements in R describing s in a neighborhood V of p contained in U.

(Think: at each point we are allowed to divide by functions which are non-vanishing near that point — rational functions which are regular on $V \ni p$.)

Def: The **spectrum of R** is the pair $(\operatorname{Spec}(R), \mathcal{O})$, and an **affine scheme** is a locally ringed space of this form.

Trigger warning for logicians: When it is clear from context that the category under consideration is, e.g., schemes or locally ringed spaces rather than topological spaces, it is convenient common practice to simply write $\operatorname{Spec}(R)$ for $(\operatorname{Spec}(R), \mathcal{O})$. We will adopt this established tradition.

Why Spec? Note that points in the classical algebraic geometry sense correspond only to maximal ideals, rather than all prime ideals. However, prime ideals interact better with morphisms: if $f: R \to S$ is a ring homomorphism and P is a prime ideal of S, then $f^{-1}(P)$ is a prime ideal of R. However, if $\mathfrak{m}$ is a maximal ideal of S, $f^{-1}(\mathfrak{m})$ need not be maximal in R.

Def: The classical algebraic geometry points – maximal ideals – are called the **closed points.**

A morphism of affine schemes $f: \operatorname{Spec}(R) \to \operatorname{Spec}(S)$ is induced by a ring homomorphism $\varphi: S \to R$. Explicitly, for $P \in \operatorname{Spec}(R)$, $f(P) = \varphi^{-1}(P)$.

Further discussion of this morphism may be found in Hartshorne Prop II.2.3.b.

Algebraic Tori

* We will always work over $\mathbb{C}$ in this course.

In reality, this says what is actually meant when a toric geometer writes "$(\mathbb{C}^*)^d$".

Def: Let $M \cong \mathbb{Z}^d$ be a rank d lattice. Then $\operatorname{Spec}(\mathbb{C}[M]) =: T_{M^*} \cong (\mathbb{C}^*)^d$ is a **d-dimensional torus**. The lattice M here is called the **character lattice of T_{M^*}**.

Note that this is a group.

Intuitively, we think of multiplication in $(\mathbb{C}^*)^d$.

More carefully, the multiplication here is a morphism of schemes$/\mathbb{C}$

$$\mu: \operatorname{Spec}(\mathbb{C}[M]) \times_{\operatorname{Spec}(\mathbb{C})} \operatorname{Spec}(\mathbb{C}[M]) = \operatorname{Spec}(\mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[M]) \longrightarrow \operatorname{Spec}(\mathbb{C}[M])$$

Fiber product. See Hartshorne II.3.

induced by:

$$\mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[M] \longleftarrow \mathbb{C}[M] : \tilde{\mu}$$
$$z^m \otimes z^m \longmapsfrom z^m$$

The $\mathbb{C}$-algebra homomorphism $\tilde{\mu}$ is called **comultiplication**.

To relate this to the intuitive idea, we consider the closed points of these schemes.

Fix a $\mathbb{Z}$-basis $\{f_1, \dots, f_d\}$ of M. Then the maximal ideals of $\mathbb{C}[M]$ have the form

$I = (z^{f_1} - c_1, \dots, z^{f_d} - c_d)$, while those of $\mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[M]$ have the form

$J = \big((z^{f_1} - a_1) \otimes 1,\ 1 \otimes (z^{f_1} - b_1),\ \dots, (z^{f_d} - a_d) \otimes 1,\ 1 \otimes (z^{f_d} - b_d) \big)$.

I claim that for J the above closed point of $\operatorname{Spec}(\mathbb{C}[M]) \times_{\operatorname{Spec}(\mathbb{C})} \operatorname{Spec}(\mathbb{C}[M])$,

$\mu(J) := \tilde{\mu}^{-1}(J) = (z^{f_1} - a_1 b_1, \dots, z^{f_d} - a_d b_d)$, which recovers the usual multiplication.

First, note that
$$\begin{aligned} \tilde{\mu}(z^{f_i} - a_i b_i) &= z^{f_i} \otimes z^{f_i} - a_i b_i \otimes 1 \\ &= (z^{f_i} - a_i) \otimes (z^{f_i} - b_i) + (z^{f_i} - a_i) \otimes b_i + a_i \otimes (z^{f_i} - b_i) \\ &\in J. \end{aligned}$$

So $\tilde{\mu}((z^{f_1} - a_1 b_1, \dots, z^{f_d} - a_d b_d)) \subset J$, and $(z^{f_1} - a_1 b_1, \dots, z^{f_d} - a_d b_d) \subset \tilde{\mu}^{-1}(J)$.

But $(z^{f_1} - a_1 b_1, \dots, z^{f_d} - a_d b_d)$ is a maximal ideal, so this containment implies equality.