Toric degenerations of cluster varieties

Timothy Magee

Instituto de Matemáticas de la Univesidad Nacional Autónoma de México Unidad Oaxaca

Joint work with Lara Bossinger, Juan Bosco Frías Medina, and Alfredo Nájera Chávez
arXiv:1809.08369 [math.AG]

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- Two classes of gluing maps, giving two types of cluster varieties: $\mathcal{A}$ and $\mathcal{X}$.

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$\mathcal{A}$-varieties

- Gluing:

$$A_k \cdot \mu_k^* \left(A'_k\right) = \prod_{j:\epsilon_{kj}>0} A_j^{\epsilon_{kj}} + \prod_{j:\epsilon_{kj}<0} A_j^{-\epsilon_{kj}}$$

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- $\mathcal{X}$ tori endowed with Poisson structure: $\{X_i, X_j\} = \epsilon_{ij} X_i X_j$.

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- We focus on $\mathcal{X}$ side.

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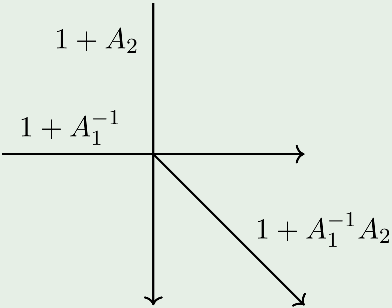

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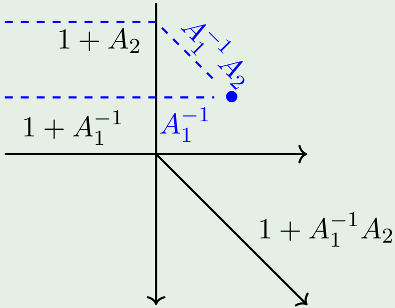

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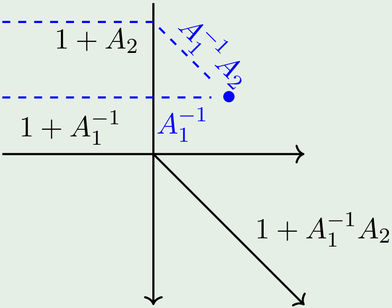

$$\mu_1^*\left(A_1'\right) = A_1^{-1} + A_1^{-1}A_2$$

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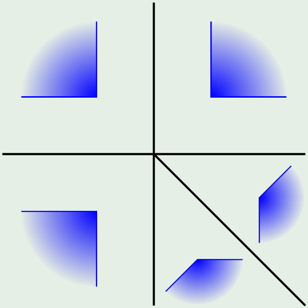

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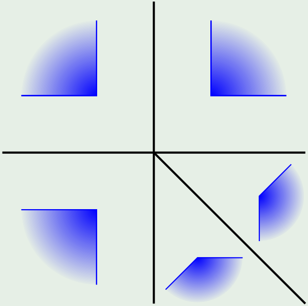

Dual cones spanned by tropical limits of $\mathcal{X}$ variables– $\mathbf{c}$-*vectors*

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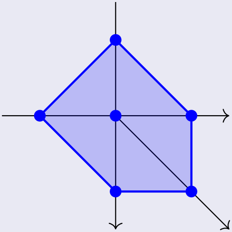

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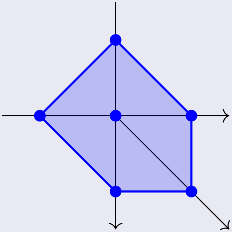

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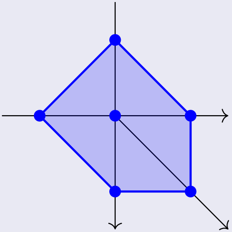

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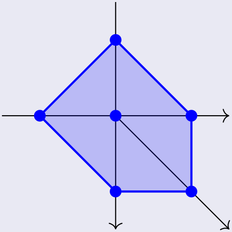

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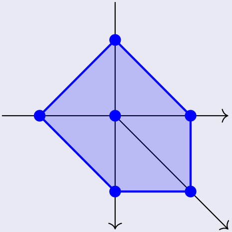

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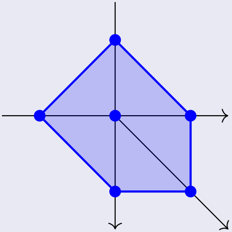

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- Result: Projective variety compactifying $\mathcal{A}$, endowed with ample line bundle $\mathcal{L}$ and basis for the section ring of $\mathcal{L}$. Just like construction of projective toric varieties by polytopes.

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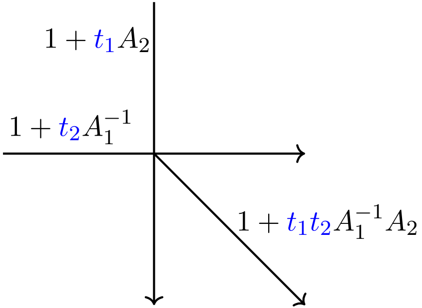

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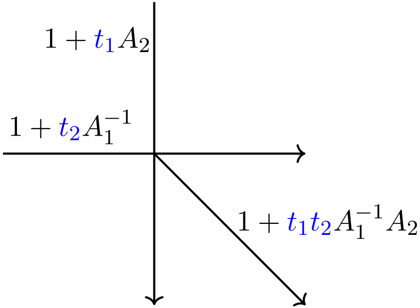

The $\mathbf{g}$-*vector* of a cluster monomial is the degree of its extension with principal coefficients.

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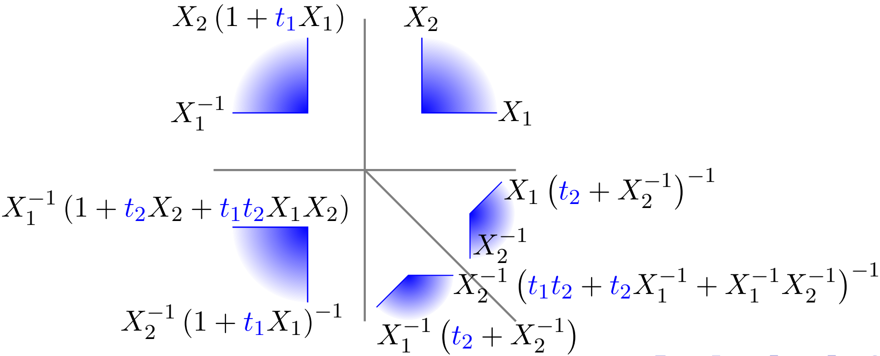

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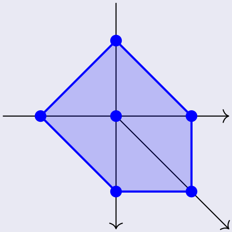

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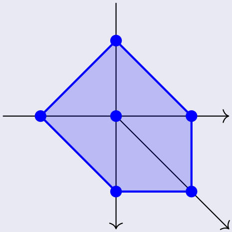

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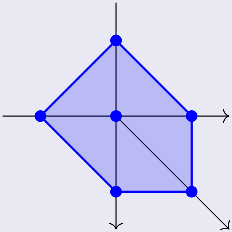

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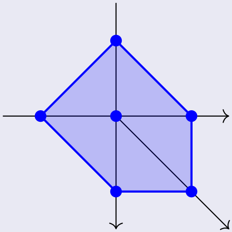

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