Math 316: Several Variable Calculus

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1 Real Vector Spaces

This will be a course on vector calculus, so our first task is to address the following question:

Question: What is a vector?

This question can be strangely divisive, and standard usage of the term doesn't quite line up with the standard definition. If you ask a room full of mathematicians for the definition of a vector, the most common answer you get will probably (and annoyingly) be:

Definition: A *vector* is an element of a vector space.

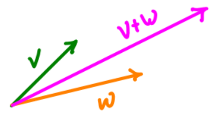
Very helpful, right? Unfortunately, this pedantic definition doesn't even cover a lot of the common use cases of the word *vector* in math. Elements of lattices are generally called vectors, and (Laurent) monomials in several variables have *exponent vectors* – this is really the same as the lattice example – but lattices are *not* vector spaces. We can cover the common usage of the word *vector* by making our pedantic definition even more annoying than before:

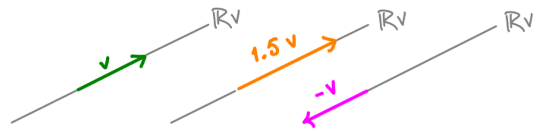
Definition: A *vector* is an element of a set with a natural inclusion into a vector space.

Any lattice (think of a grid) will naturally include into real vector space, so this annoying addition will address that gripe. But we had better say what a vector space is now. For simplicity, we will only discuss *real vector spaces*. That is, we will work over the real numbers $\mathbb{R}$. There are plenty of other perfectly good number systems, or more accurately *fields*, we could work over, such as the rational numbers $\mathbb{Q}$ or complex numbers $\mathbb{C}$. But defining a *field* would be a bit of a detour. Just be aware that what we are about to discuss fits into a broader picture. The structure would remain the same if we replaced $\mathbb{R}$ with any other field.

We'll give a rough idea first, and then a more careful definition.

Idea: A real vector space is a space where we can add points together and rescale points by any real number.





Careful version: A *real vector space* is a set V with an addition operation

$$\begin{aligned} + : V \times V &\to V \\ (\mathbf{v}, \mathbf{w}) &\mapsto \mathbf{v} + \mathbf{w} \end{aligned}$$

and a real scalar multiplication

$$\begin{aligned} \cdot : \mathbb{R} \times V &\to V \\ (r, \mathbf{v}) &\mapsto r \cdot \mathbf{v} \end{aligned}$$

satisfying the following conditions. For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $r, s \in \mathbb{R}$:

- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
  (Associativity of Vector Addition)
- $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$
  (Commutativity of Vector Addition)
- There is an element $\mathbf{0} \in V$ with $\mathbf{0} + \mathbf{v} = \mathbf{v}$.
  (Identity Element of Vector Addition)
- There is an element $-\mathbf{v} \in V$ with $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
  (Inverse Elements of Vector Addition)
- $r \cdot (s \cdot \mathbf{v}) = (rs) \cdot \mathbf{v}$
  (Compatibility of Scalar Multiplication with Multiplication in $\mathbb{R}$)
- For the element $1 \in \mathbb{R}$, we have $1 \cdot \mathbf{v} = \mathbf{v}$.
  (Identity Element of Scalar Multiplication)
- $r \cdot (\mathbf{v} + \mathbf{w}) = r \cdot \mathbf{v} + r \cdot \mathbf{w}$
  (Distributivity of Scalar Multiplication with Respect to Vector Addition)
- $(r + s) \cdot \mathbf{v} = r \cdot \mathbf{v} + s \cdot \mathbf{v}$
  (Distributivity of Scalar Multiplication with Respect to Addition in $\mathbb{R}$)

Notation 1. Usually, we will just write $r\mathbf{v}$ for $r \cdot \mathbf{v}$.

Question 2. Does $\mathbb{R}$ itself have the structure of a vector space?

Question 3. What about $\mathbb{R}^3$ with head-to-tail addition?

The example we've just seen is a very common way to picture vectors. In fact, if you take an introductory physics class, or consult the textbook for this class, you'll see the definition: *A vector is a quantity that has both magnitude and direction.* This is a nice intuitive notion to keep in mind, but it won't be quite sufficient for our needs in this course.

Question 4. Do you think there is anything special about dimension 3? Would anything change if we replaced $\mathbb{R}^3$ with $\mathbb{R}^n$?

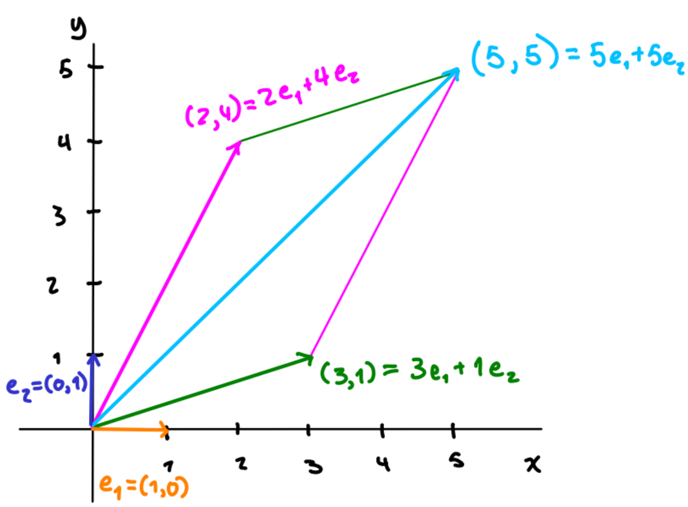
It's easy to imagine reducing the dimension from 3 to 2, 1, or even 0. Things are a bit trickier if we want to increase the dimension though. A convenient way to make sense of this is to equip $\mathbb{R}^n$ with a *basis*.

Idea: A *basis* of V is a subset $B \subset V$ such that every element $\mathbf{v} \in V$ can be written as a finite $\mathbb{R}$-linear combination of elements of B in exactly one way.

Careful version: A *basis* B of a real vector space V is a subset of V satisfying:

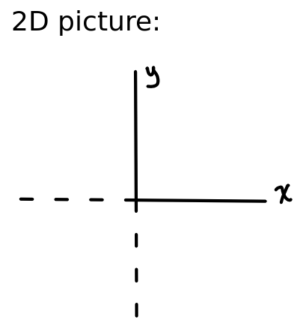
- for every finite subset $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ of B, we can only have an equality $c_1\mathbf{e}_1 + \cdots + c_m\mathbf{e}_m = \mathbf{0}$ if $c_1 = \cdots = c_m = 0$.
(Linear Independence)
- for every $\mathbf{v} \in V$, there is a finite subset $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ of B and associated scalars $c_1, \ldots, c_m \in \mathbb{R}$ such that $v = c_1\mathbf{e}_1 + \cdots + c_m\mathbf{e}_m$.
(Spanning Property)

In this way, head-to-tail addition becomes simply componentwise addition.



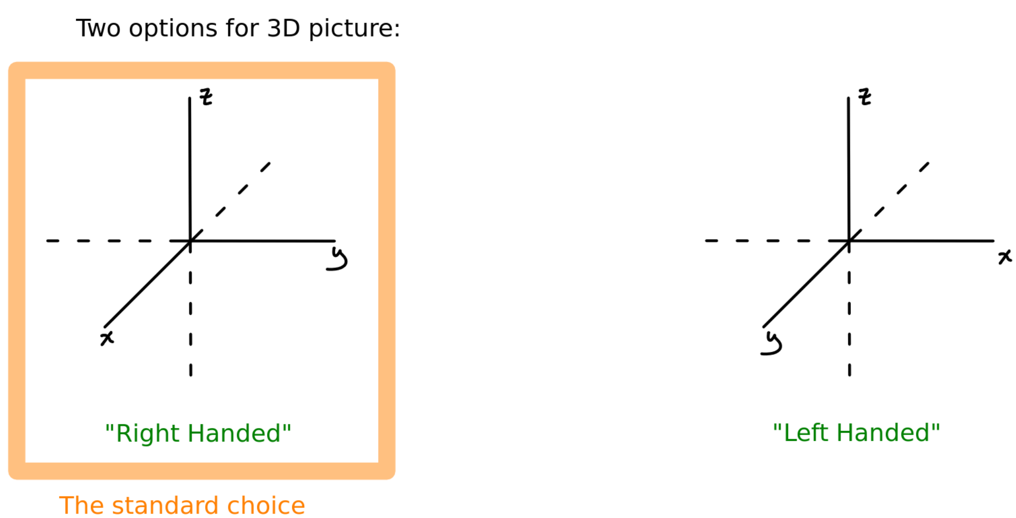
When we express vectors in this way, we are using a *Cartesian coordinate system.* Let's try drawing both 2 and 3 dimensional Cartesian coordinate systems. You've seen the standard

2 dimensional Cartesian coordinate system plenty of times in your mathematical career by now.

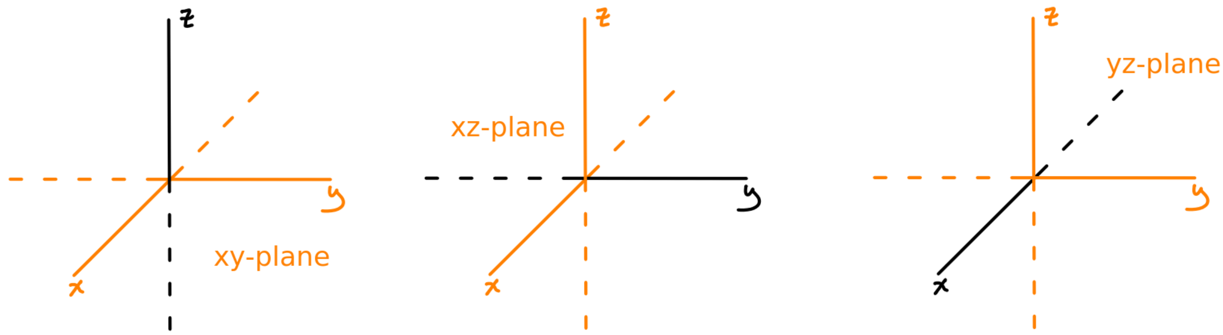


Of course, we've made a choice here. We could just as well have chosen the x-axis to be vertical and the y-axis to be horizontal. It's just standard convention to draw it the way we've shown. The two options are different *orientations*, and up to a certain notion of equivalence, these really are the only possibilities for our Cartesian coordinate system. We will make this a bit more precise in a few weeks.

Now let's try three dimensions. We again have two options up to our still-unexplained notion of equivalence.



In 2D we have coordinate axes: the x-axis and y-axis. We can project onto these two axes from our coordinate plane. In 3D, we still have these (and the z-axis), but we have projections onto coordinate planes as well: the xy-plane, xz-plane, and yz-plane.

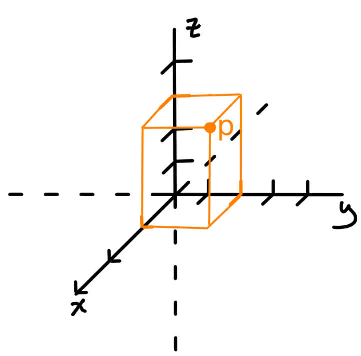


Bonus Question: *(For 10 points on Homework 1)* Consider a Cartesian coordinate system for d-space, where d is some positive integer. The coordinate subspaces are the subspaces of d-space spanned by some collection of coordinate axes. How many positive dimensional proper coordinate subspaces are there? In other words, count the coordinate subspaces of d-space, excluding 0 and d-space itself.

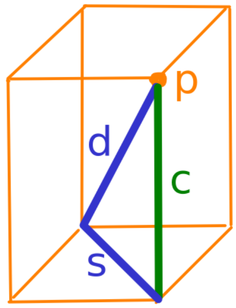
When we are working in a three dimensional Cartesian coordinate system, we write points as $(x-\text{coordinate}, y-\text{coordinate}, z-\text{coordinate})$.

Example 5.

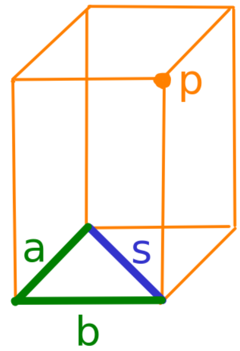
$$p = (1, 2, 3)$$



Question 6. How far is $p = (a, b, c)$ from the origin? Is there some way to reduce this question to the 2 dimensional Pythagorean theorem?



$d = \sqrt{s^2 + c^2}$



$s = \sqrt{a^2 + b^2}$

$$\implies d = \sqrt{a^2 + b^2 + c^2}$$

Notation 7. If we have a vector $\mathbf{v}$ in the "magnitude and direction" sense, we denote the magnitude by $\|\mathbf{v}\|$. This is also called the *norm* of $\mathbf{v}$. We call a vector of norm 1 a *unit vector*, and we denote the unit vector with the same direction as $\mathbf{v}$ by $\widehat{\mathbf{v}}$.

Question 8. If $\mathbf{v}$ is a non-zero vector of the form described above, can you express $\widehat{\mathbf{v}}$ in terms of $\mathbf{v}$ and $\|\mathbf{v}\|$?

The unit vector $\widehat{\mathbf{v}}$ is called the *normalization* of $\mathbf{v}$.

Example 9. Let's work in 2-space, equipped with the standard Cartesian coordinate system – the x-axis generated by $\mathbf{e}_1$ and the y-axis generated by $\mathbf{e}_2$. Now consider a change of coordinates, such that the positive x'-axis is spanned by $\mathbf{e}_1 + \mathbf{e}_2$ and the positive y'-axis is spanned by $-\mathbf{e}_1 + \mathbf{e}_2$. What are the unit vectors $\mathbf{e}_1'$ and $\mathbf{e}_2'$ in the positive x'- and y'-axes?

2 Real Affine Spaces

So far we have treated $\mathbb{R}^n$ as a real vector space, and we have treated points in $\mathbb{R}^n$ as vectors. If you've been reading the textbook, you may have noticed that the authors are careful to distinguish between points in $\mathbb{R}^n$ and vectors (also in $\mathbb{R}^n$). In fact, the set $\mathbb{R}^n$ can be equipped with many different algebraic structures. When the authors distinguish between points and vectors, they are viewing the copy of $\mathbb{R}^n$ where the points live as an *affine space* and the copy of $\mathbb{R}^n$ where the vectors live as the *associated vector space*. Let's call our affine space A and the associated vector space V. As sets, A and V can be identified. (That is, we can give a *bijection* $f : A \to V$. This means there is a map $g : V \to A$ with $f \circ g = \mathrm{id}_V$ and $g \circ f = \mathrm{id}_A$. You'll explore this concept in more detail in a couple weeks if you are taking *Transition to Advanced Mathematics* this semester.) But we don't necessarily have a notion of addition

$$+ : A \times A \to A$$

or scalar multiplication

$$\cdot : \mathbb{R} \times A \to A.$$

Instead, we have another notion of addition, which we brazenly denote with the exact same symbol:

$$\begin{aligned} + : A \times V &\to A \\ (a, \mathbf{v}) &\mapsto a + \mathbf{v}. \end{aligned}$$

This addition is required to satisfy three properties.

- *(Right identity)* For all $a \in A$, $a + \mathbf{0} = a$.
- *(Associativity)* For all $a \in A$ and $\mathbf{v}, \mathbf{w} \in V$, $(a + \mathbf{v}) + \mathbf{w} = a + (\mathbf{v} + \mathbf{w})$.
- For each $a \in A$, the map

  $$\begin{aligned} V &\to A \\ v &\mapsto a + \mathbf{v} \end{aligned}$$

  is a bijection.

Note that there may not be a $\mathbf{0}$ element – an additive identity element – in A. (There may not even be an addition operation on A where it would make sense to talk about an additive identity in A.) As such, affine spaces may not have distinguished points. Each point is on exactly the same footing.

Let's see an example of an affine space that is *not* a vector space.

Example 10. Let A be the plane in $\mathbb{R}^3$ defined by the equation $z = 1$. If we were to try to add two points a_1 and a_2 in A, the result $a_1 + a_2$ would lie in the plane $z = 2$. So we can't add a pair of points in A and remain in A.

Can you describe the associated vector space V in this case?

Now let $a = (3, 2, 1)$. Can you describe the map

$$\begin{aligned} V &\to A \\ \mathbf{v} &\mapsto a + \mathbf{v}? \end{aligned}$$

We can use the addition of points in affine space with vectors in the associated vector space to define a subtraction

$$\begin{aligned} A \times A &\to V \\ (a_2, a_1) &\mapsto a_2 - a_1. \end{aligned}$$

Here, $\mathbf{v} = a_2 - a_1$ is the unique vector satisfying $a_1 + \mathbf{v} = a_2$.

Question 11. Let A be the affine space of the previous example. If $a_1 = (3, 2, 1)$ and $a_2 = (-1, -2, 1)$, what is $a_2 - a_1$? Can you provide a sketch to accompany this calculation?

The authors of the textbook differentiate between points in affine space and vectors in the associated vector space notationally – they use parentheses for points in affine space, *e.g.* (x, y, z), and angle brackets for vectors in the associated vector space, *e.g.* $\langle x, y, z \rangle$.

3 "Products" of Vectors

We've said earlier that we can add vectors together, or multiply a vector by a scalar, but we don't have a way to multiply two vectors. It's time to change that. We'll talk about two different sorts of products of vectors. The first will be scalar-valued. That is, we will have a map $V \times V \to \mathbb{R}$. The next will in a sense be vector-valued, but – and this point is crucial – *not* valued in the *same* vector space. It will be a map $V \times V \to W$, where W is some other vector space that we will describe in a bit.

3.a Inner Products

The type of scalar-valued product we will discuss is called an *inner product*. You should be aware that not every vector space comes equipped with an inner product. Vector spaces that do come equipped with an inner product are called *inner product spaces*.

Definition 12. An *inner product* on a real vector space V is a map

$$\langle \,\cdot\, , \,\cdot\, \rangle : V \times V \to \mathbb{R}$$

satisfying:

- *(Symmetry)* $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$ for all $\mathbf{v}, \mathbf{w} \in V$.
- *(Linearity)* $\langle a\mathbf{u} + b\mathbf{v}, \mathbf{w} \rangle = a\langle \mathbf{u}, \mathbf{w} \rangle + b\langle \mathbf{v}, \mathbf{w} \rangle$ for all $a, b \in \mathbb{R}$, $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.
- *(Positive Definiteness)* $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ for all non-zero $\mathbf{v} \in V$.

Example 13. Let's consider a real vector space V of the "magnitude and direction" sort. Then given a pair of vectors $\mathbf{v}, \mathbf{w} \in V$, we can talk about the magnitude $\|\mathbf{v}\|$ and $\|\mathbf{w}\|$ of each, as well as the angle $\theta_{\mathbf{v},\mathbf{w}}$ between their directions if both are non-zero. So, we can define a map

$$\begin{aligned} &V \times V \to \mathbb{R} \\ &(\mathbf{v}, \mathbf{w}) \mapsto \begin{cases} \|\mathbf{v}\|\|\mathbf{w}\| \cos(\theta_{\mathbf{v},\mathbf{w}}) & \text{if } \mathbf{v} \text{ and } \mathbf{w} \text{ both non-zero} \\ 0 & \text{otherwise.}^1 \end{cases} \end{aligned}$$

Symmetry and positive definiteness of this map are almost immediate. Since cos is an even function (meaning $\cos(-\theta) = \cos(\theta)$), $\cos(\theta_{\mathbf{v},\mathbf{w}}) = \cos(\theta_{\mathbf{w},\mathbf{v}})$, from which we obtain symmetry. Next, $\langle \mathbf{v}, \mathbf{v} \rangle = \|\mathbf{v}\|^2 \cos(0) = \|\mathbf{v}\|^2$. This is positive unless $\mathbf{v} = \mathbf{0}$.

For linearity, we want to show that

$$\langle a\mathbf{u} + b\mathbf{v}, \mathbf{w} \rangle \stackrel{?}{=} a\langle \mathbf{u}, \mathbf{w} \rangle + b\langle \mathbf{v}, \mathbf{w} \rangle.$$

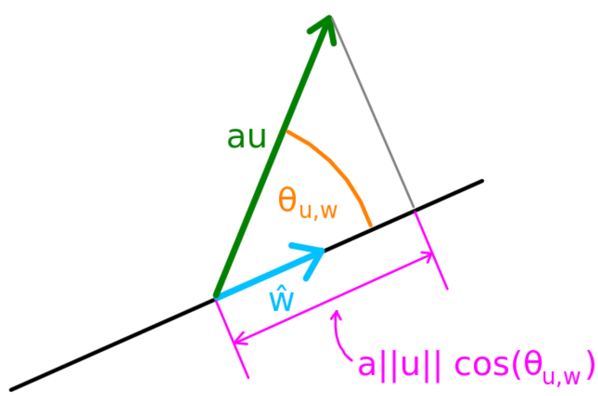
This is clear if $\mathbf{u}$, $\mathbf{v}$, or $\mathbf{w}$ is $\mathbf{0}$, or if a or b is 0. So, let's assume each of these is non-zero. Then we want to show

$$\begin{aligned} \langle a\mathbf{u} + b\mathbf{v}, \mathbf{w} \rangle :=& \|a\mathbf{u} + b\mathbf{v}\| \|\mathbf{w}\| \cos\left(\theta_{\mathbf{u}+\mathbf{v},\mathbf{w}}\right) \\ \stackrel{?}{=}& a\|\mathbf{u}\| \|\mathbf{w}\| \cos\left(\theta_{\mathbf{u},\mathbf{w}}\right) + b\|\mathbf{v}\| \|\mathbf{w}\| \cos\left(\theta_{\mathbf{v},\mathbf{w}}\right) \\ =:& \, a\langle \mathbf{u}, \mathbf{w} \rangle + b\langle \mathbf{v}, \mathbf{w} \rangle. \end{aligned}$$

Since $\|\mathbf{w}\| \neq 0$, we can divide both sides by $\|\mathbf{w}\|$. We have the equality if and only if

$$\|a\mathbf{u} + b\mathbf{v}\| \cos\left(\theta_{\mathbf{u}+\mathbf{v},\mathbf{w}}\right) \stackrel{?}{=} a\|\mathbf{u}\| \cos\left(\theta_{\mathbf{u},\mathbf{w}}\right) + b\|\mathbf{v}\| \cos\left(\theta_{\mathbf{v},\mathbf{w}}\right). \tag{1}$$

Note that $\|a\mathbf{u}+b\mathbf{v}\| \cos\left(\theta_{\mathbf{u}+\mathbf{v},\mathbf{w}}\right)$ is the projection of $a\mathbf{u}+b\mathbf{v}$ onto the span of $\mathbf{w}$, $a\|\mathbf{u}\| \cos\left(\theta_{\mathbf{u},\mathbf{w}}\right)$ is the projection of $a\mathbf{u}$ onto this line, and $b\|\mathbf{v}\| \cos\left(\theta_{\mathbf{v},\mathbf{w}}\right)$ is the span of $b\mathbf{v}$ onto this line. For instance, see the picture below.



So, (1) is just the $\widehat{\mathbf{w}}$ component equality

$$(a\mathbf{u} + b\mathbf{v})_{\widehat{\mathbf{w}}} = a\mathbf{u}_{\widehat{\mathbf{w}}} + b\mathbf{v}_{\widehat{\mathbf{w}}},$$

which is clear.

The inner product in the previous example has a special name and notation. It's called the *dot product* and we write $\langle \mathbf{v}, \mathbf{w} \rangle$ as $\mathbf{v} \cdot \mathbf{w}$. The description we gave in the example is a coordinate free description. More often than not, it is defined in a way that references coordinates:

[1] I'm splitting this into cases simply because $\theta_{\mathbf{v},\mathbf{w}}$ is only defined if both $\mathbf{v}$ and $\mathbf{w}$ are non-zero. Of course, the previous line would evaluate to 0 for *any* possible value of $\theta_{\mathbf{v},\mathbf{w}}$ if either $\mathbf{v}$ or $\mathbf{w}$ were $\mathbf{0}$.

Definition 14. The *dot product* on $\mathbb{R}^n$ is the map

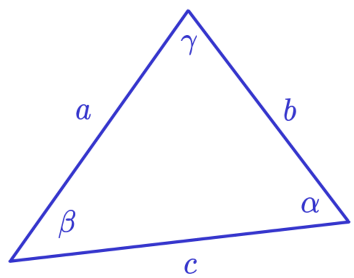
$$\begin{aligned} \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R} \\ ((a_1, \ldots, a_n), (b_1, \ldots, b_n)) &\mapsto a_1 b_1 + \cdots + a_n b_n. \end{aligned}$$

These two descriptions are related in Theorem 2.4 of the text, and we will go through that in a bit. But first –

Question 15. If we define the dot product using coordinates as in Definition 14, does the dot product satisfy the properties of an inner product?

To relate the two descriptions, we use the Law of Cosines.

Lemma (Law of Cosines)**.** *Consider a triangle with side lengths and angles labeled as follows.*



Then

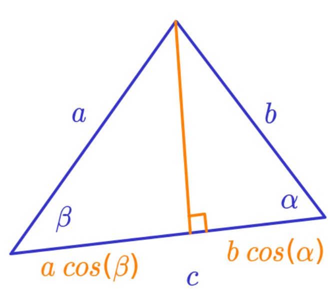
$$c^2 = a^2 + b^2 - 2ab\cos(\gamma),$$

$$b^2 = a^2 + c^2 - 2ac\cos(\beta),$$

and

$$a^2 = b^2 + c^2 - 2bc\cos(\alpha).$$

Proof. Observe that $c = a\cos(\beta) + b\cos(\alpha)$.



Similarly, $b = a\cos(\gamma) + c\cos(\alpha)$ and $a = b\cos(\gamma) + c\cos(\beta)$. Then

$$c^2 = ac\cos(\beta) + bc\cos(\alpha),$$

$$b^2 = ab\cos(\gamma) + bc\cos(\alpha),$$

and

$$a^2 = ab\cos(\gamma) + ac\cos(\beta).$$

So,

$$c^2 - a^2 - b^2 = ac\cos(\beta) + bc\cos(\alpha) - ab\cos(\gamma) - bc\cos(\alpha) - ab\cos(\gamma) - ac\cos(\beta),$$

and

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma).$$

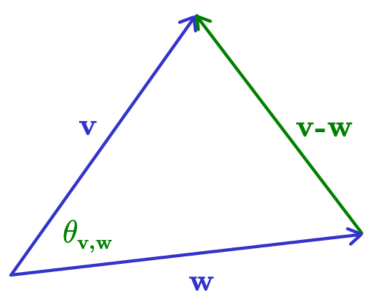
The other equations are proved analogously. □

Proposition 16. *The two descriptions of the dot product agree. That is, if both* $\mathbf{v}$ *and* $\mathbf{w}$ *are non-zero,*

$$\|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta_{\mathbf{v},\mathbf{w}}) = v_1w_1 + \cdots + v_nw_n.$$

(Agreement is clear if either is $\mathbf{0}$.)

Proof. Consider the triangle below.



Note that

$$\begin{aligned}\|\mathbf{v}-\mathbf{w}\|^2 &= (v_1 - w_1)^2 + \cdots + (v_n - w_n)^2 \\ &= \left(v_1^2 - 2v_1w_1 + w_1^2\right) + \cdots + \left(v_n^2 - 2v_nw_n + w_n^2\right) \\ &= \left(v_1^2 + \cdots + v_n^2\right) + \left(w_1^2 + \cdots + w_n^2\right) - 2\left(v_1w_1 + \cdots + v_nw_n\right) \\ &= \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\left(v_1w_1 + \cdots + v_nw_n\right).\end{aligned} \tag{2}$$

Next, using the Law of Cosines,

$$\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\| \cos(\theta_{\mathbf{v},\mathbf{w}}). \tag{3}$$

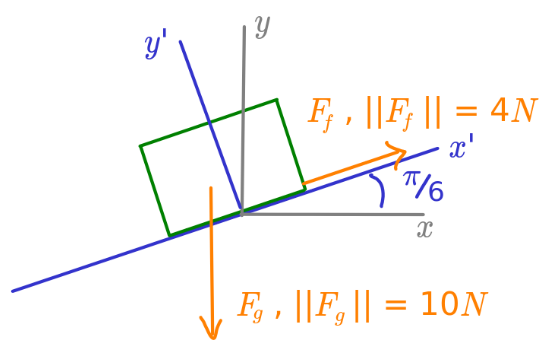
Comparing Equations (2) and (3), we find that

$$\|\mathbf{v}\|\|\mathbf{w}\| \cos(\theta_{\mathbf{v},\mathbf{w}}) = v_1 w_1 + \cdots + v_n w_n.$$

$\square$

We can apply Proposition 16 to describe projections of vectors onto different coordinate axes. As we saw in Example 13, $\mathbf{v} \cdot \mathbf{w}$ is the projection of $\mathbf{v}$ onto the axis spanned by $\mathbf{w}$. This let's us decompose vectors into components easily, even when we want to express those components in a different coordinate system than the one we started in. This can be very handy when decomposing forces in physics, for example.

Example 17. Consider the following force diagram for a sliding down a ramp.



What are the unit vectors $\mathbf{e}_1'$ and $\mathbf{e}_2'$ in terms of $\mathbf{e}_1$ and $\mathbf{e}_2$? Express the forces in the new coordinate system. Why might you want to do this?

We want to write $\mathbf{e}_1' = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ and $\mathbf{e}_2' = b_1\mathbf{e}_1 + b_2\mathbf{e}_2$ for some real coefficients a_1, a_2, b_1, and b_2. If pr_1 and pr_2 denote the projections onto the x and y axes, then we have

$$\mathrm{pr}_1(\mathbf{e}_1') = a_1\mathbf{e}_1,$$

$$\mathrm{pr}_2(\mathbf{e}_1') = a_2\mathbf{e}_2,$$

$$\mathrm{pr}_1(\mathbf{e}_2') = b_1\mathbf{e}_1,$$

and

$$\mathrm{pr}_2(\mathbf{e}_2') = b_2\mathbf{e}_2.$$

So,

$$a_1 = \mathbf{e}_1' \cdot \mathbf{e}_1 = \|\mathbf{e}_1'\|\|\mathbf{e}_1\| \cos(\theta_{\mathbf{e}_1',\mathbf{e}_1}) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2},$$

$$a_2 = \mathbf{e}_1' \cdot \mathbf{e}_2 = \|\mathbf{e}_1'\|\|\mathbf{e}_2\| \cos(\theta_{\mathbf{e}_1',\mathbf{e}_2}) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2},$$

$$b_1 = \mathbf{e}_2' \cdot \mathbf{e}_1 = \|\mathbf{e}_2'\|\|\mathbf{e}_1\| \cos(\theta_{\mathbf{e}_2',\mathbf{e}_1}) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2},$$

and

$$b_2 = \mathbf{e}_2' \cdot \mathbf{e}_2 = \|\mathbf{e}_2'\|\|\mathbf{e}_2\| \cos(\theta_{\mathbf{e}_2',\mathbf{e}_2}) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}.$$

Next,

$$\mathbf{F}_g \cdot \mathbf{e}_1' = \|\mathbf{F}_g\|\|\mathbf{e}_1'\| \cos\left(\frac{2\pi}{3}\right) = -5N,$$

and

$$\mathbf{F}_g \cdot \mathbf{e}_2' = \|\mathbf{F}_g\|\|\mathbf{e}_2'\| \cos\left(\frac{5\pi}{6}\right) = -5\sqrt{3}N,$$

so $\mathbf{F}_g = -5N\mathbf{e}_1' - 5\sqrt{3}N\mathbf{e}_2'$. The frictional force is entirely in the positive x' direction, so $\mathbf{F}_f = 4N$.

This analysis would allow us to compute the acceleration of the block using Newton's 2^{nd} law of motion: $\mathbf{F} = m\mathbf{a}$. We know that the acceleration must be in the $-\mathbf{e}_1'$ direction if it accelerating down the ramp, but let's complete the analysis carefully. There is a force acting on the block that wasn't illustrated in the previous image – the *normal force* $\mathbf{F}_n$ of the ramp pushing the block in the $\mathbf{e}_2'$ direction. This will exactly counter the y'-component of the gravitational force (so the block doesn't either fly off the ramp or fall though it). So, $\mathbf{F}_n = 5\sqrt{3}N\mathbf{e}_2'$. If we add all of the forces acting on the block, we get a net force of

$$\mathbf{F}_{\text{net}} = \mathbf{F}_g + \mathbf{F}_f + \mathbf{F}_n = -\mathbf{e}_1'.$$

We can compute the mass of the block from the gravitational force: $\mathbf{F}_g = m\mathbf{g}$, where $\mathbf{g}$ is the acceleration due to gravity on Earth's surface. This is about $10\frac{\text{m}}{\text{s}^2}$ toward the center of the earth – in the direction of $-\mathbf{e}_2$ in this example. So, the mass of the block is $1\frac{N\text{s}^2}{\text{m}}$, which is a silly way of writing 1kg. Then the acceleration is $-1\frac{\text{m}}{\text{s}^2}\mathbf{e}_1'$.

A few words of interpretation before moving on to vector-valued products. The dot product of $\mathbf{v}$ and $\mathbf{w}$ is a real number that can take values between $-\|\mathbf{v}\|\|\mathbf{w}\|$ (if the two vectors are pointed in exactly opposite directions) and $\|\mathbf{v}\|\|\mathbf{w}\|$ (if the two vectors are pointed the same direction). Right between these two extremes, we have the case in which the pair of vectors form a right angle. In this case their dot product is zero. In fact, we will use this to describe when two vectors are orthogonal to each other, and this applies in *any* inner product space:

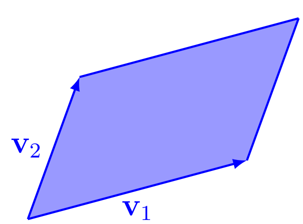
Definition 18. Let $(V, \langle \cdot\,, \cdot \rangle)$ be an inner product space, and let $\mathbf{v}$ and $\mathbf{w}$ be vectors in V. We say $\mathbf{v}$ is *orthogonal* to $\mathbf{w}$ if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.

Relatedly, the bases we have considered so far have all consisted of unit vectors which were orthogonal to one another. Note that this isn't required in the definition of a basis – but these are certainly convenient properties for a basis to have.

Definition 19. Let $(V, \langle \cdot\,, \cdot \rangle)$ be an inner product space, and let B be a basis for V. We say B is an *orthonormal basis* if $\langle \mathbf{v}, \mathbf{v} \rangle = 1$ for every $\mathbf{v} \in B$ and $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ for every pair of distinct basis vectors $\mathbf{v}$ and $\mathbf{w}$ in B.

3.b Wedge Products

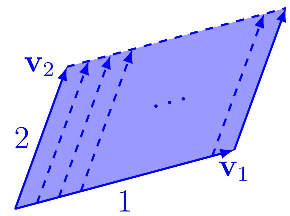
The next version of a vector product we will discuss is intimately related to areas, volumes, and their higher dimensional analogues (called k-volumes in dimension k). In fact, if V is a vector space of the "magnitude and direction" sort, we can interpret a *wedge product* of two vectors $\mathbf{v}_1 \wedge \mathbf{v}_2$ as an oriented area element. Specifically, it is the area of the parallelogram defined by the pair of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, and if we swap the order we introduce a sign. That is, $\mathbf{v}_1 \wedge \mathbf{v}_2$ and $\mathbf{v}_2 \wedge \mathbf{v}_1$ both represent the area shown below, but one is the negative of the other as they correspond to opposite orientations of this parallelogram.



Similarly, the wedge product of three vectors $\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \mathbf{v}_3$ is the oriented volume of the parallelepiped defined by the three vectors. More generally, $\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k$ is the oriented k-volume of the k-dimensional parallelotope (this means exactly what you would expect it to) defined by the k vectors.

Note that I am specifically referring to the *area* of the parallelogram, *volume* of the parallelepiped, and *k-volume* of the parallelotope. I am *not* saying the wedge product *is* the parallelogram, the parallelepiped, or the parallelotope. If $\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k$ and $\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_k$ define the same oriented k-volume – even though they may correspond to different parallelotopes – then we will have $\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k = \mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_k$.

Before trying to work out the right way to encapsulate this notion in a mathematical definition, let's talk a bit about orientation. We'll start in dimension 2. Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ define a parallelogram of non-zero area, *i.e.* $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. We'll picture the construction of the parallelogram by first placing $\mathbf{v}_1$ and labeling this edge with a 1, then placing a copy of $\mathbf{v}_2$ at every point along the length of $\mathbf{v}_1$. Label the edge meeting the tail of $\mathbf{v}_1$ with a 2.



Let's call the result a *labeled parallelogram*, and denote it $P(\mathbf{v}_1, \mathbf{v}_2)$. This has more information than just a parallelogram because it keeps a record of how it was constructed. There are many operations we can perform on our labeled parallelogram that will preserve its area – all compositions of rotations, translations, shears,[2] and reflections. In fact, a labeled parallelogram can be transformed into any other labeled parallelogram of the same area with some composition of these four types of transformations. Notice that the first three of these can be performed continuously, but reflections involve a discrete jump. If we also allow scaling by positive constants λ (which would rescale the area by λ^2), we could transform our labeled parallelogram into *any* labeled parallelogram of non-zero area.

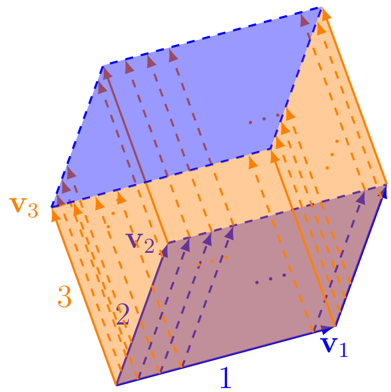
Question 20. In class Missy brought up an interesting idea for an area preserving transformation that is not a rotation, translation, shear, or reflection. She suggested expanding in one direction and shrinking in another. For example, scale all x-coordinates by λ and y-coordinates by $\frac{1}{\lambda}$. This would indeed send parallelograms to parallelograms of the same area. So, do we need to add to our previous list of transformations, or can we express this as a composition of rotations, translations, shears, and reflections?

Definition 21. Let V be a 2 dimensional real vector space of the "magnitude and direction" sort. We say two labeled parallelograms of non-zero area in V have the *same orientation* if one can be continuously deformed into the other without ever leaving the world of labeled parallelograms of non-zero area. Otherwise we say the labeled parallelograms have *opposite orientations.*

Question 22. Do $P(\mathbf{v}_1, \mathbf{v}_2)$ and $P(\mathbf{v}_2, \mathbf{v}_1)$ have the same orientation or opposite orientations? What about $P(\mathbf{v}_1, \mathbf{v}_2)$ and $P(-\mathbf{v}_1, -\mathbf{v}_2)$?

Now we move up to dimension 3. In this case we will will treat *labeled parallelepipeds.* These are defined by an ordered linearly independent collection of three vectors $(\mathbf{v}_1, \mathbf{v}_2.\mathbf{v}_3)$. The first two vectors define the labeled parallelogram $P(\mathbf{v}_1, \mathbf{v}_2)$. To construct the labeled parallelepiped $P(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, we affix a copy of $\mathbf{v}_3$ to every point of $P(\mathbf{v}_1, \mathbf{v}_2)$.

[2]For instance, adding any multiple of $\mathbf{v}_1$ to $\mathbf{v}_2$.



Labeled parallelepipeds of the same non-zero volume can also be related by rotations, translations, shears, and reflections. Again, the first three can be done continuously while reflections require a discrete jump. And if we also allow scaling by positive constants λ (which would rescale the volume by λ^3), we could transform our labeled parallelepiped into *any* labeled parallelepiped of non-zero volume.

Definition 23. Let V be a 3 dimensional real vector space of the "magnitude and direction" sort. We say two labeled parallelepipeds of non-zero volume in V have the *same orientation* if one can be continuously deformed into the other without ever leaving the world of labeled parallelepipeds of non-zero volume. Otherwise we say the labeled parallelepipeds have *opposite orientations*.

Question 24. Let σ be any permutation of $(1, 2, 3)$. When do $P(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $P(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \mathbf{v}_{\sigma(3)})$ have the same orientation, and when do they have opposite orientations?

$P(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $P(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \mathbf{v}_{\sigma(3)})$ have the same orientation if and only if σ is an *even* permutation, meaning it can be decomposed into an even number of transpositions.

What we see in dimension 3 extends directly to dimension k.

Definition 25. Let V be a k dimensional real vector space of the "magnitude and direction" sort. We say two labeled k-parallelotopes of non-zero k-volume in V have the *same orientation* if one can be continuously deformed into the other without ever leaving the world of labeled k-parallelotopes of non-zero k-volume. Otherwise we say the labeled k-parallelotopes have *opposite orientations*.

If you've taken the Algebra course already, you may have come across the following fact:

Fact. *The even permutations are generated by 3-cycles.*

Note that the 3-cycles you considered in Question 24 corresponded to rotations in a way that can be extended to higher dimension – just fix the orthogonal complement of the three vectors under consideration. So the k-dimensional picture reduces to the 3 dimensional picture.

One last point of clarification on orientation. Note that we have required the ambient vector space to have the same dimension as the parallelotope in order to define its orientation. So, if we have a pair of parallelograms in 3-space for instance, we can't in general say if they have the same orientation. In this case, comparing orientations will only make sense if the parallelograms live in the same 2 dimensional subspace, or more generally if the k-parallelotopes live in the same k dimensional subspace.

So, now that we understand the idea of the oriented k-volume of a k-parallelotope, and we know that this notion is what the wedge product should capture, what properties should we ask this wedge product to have? Here are some that come to mind.

- Linearity in each argument. If we double the length of one side of a box, we double its volume.
- Shear transformations shouldn't affect it.
- $\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k$ is non-zero if and only if $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly independent set of vectors. In other words, if our box isn't squished flat, it has some non-zero volume. If it is squished flat, it has zero volume.

And that really covers it. Let's try to express this a bit more precisely now.

Definition 26. Let V be a real vector space. The k^{th} *wedge product* on V is the associative $\bigwedge^k V$-valued product

$$\begin{aligned} \wedge : V \times \cdots \times V &\to \bigwedge^k V \\ (\mathbf{v}_1, \ldots, \mathbf{v}_k) &\mapsto \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k, \end{aligned}$$

where $\bigwedge^k$ is the vector space generated by vectors of the form $\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k$, subject to the relations

- *(Alternating)* If we have $\mathbf{v}_i = \mathbf{v}_j$ for some i and j with $i \neq j$,
$$\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k = \mathbf{0}_{\bigwedge^k V}.$$
- *(Multilinear)* For any i,
$$\begin{aligned} \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{i-1} \wedge (a\mathbf{v}_i + b\mathbf{v}_i') \wedge \mathbf{v}_{i+1} \wedge \cdots \wedge \mathbf{v}_k =& a\left(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{i-1} \wedge \mathbf{v}_i \wedge \mathbf{v}_{i+1} \wedge \cdots \wedge \mathbf{v}_k\right) \\ &+ b\left(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{i-1} \wedge \mathbf{v}_i' \wedge \mathbf{v}_{i+1} \wedge \cdots \wedge \mathbf{v}_k\right). \end{aligned}$$

Question 27. Does this have the properties we asked for?

Note that in this definition we have only asked for V to be a real vector space – we haven't asked it to be a vector space of the "magnitude and direction" sort.

Question 28. Suppose V is a vector space of the "magnitude and direction" sort. Is $\bigwedge^k V$?

If V is a vector space of the "magnitude and direction" sort, we want to interpret unit k-cubes as having k-volume 1. So, we will interpret $\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}$ as having magnitude 1 as long as all indices are distinct.

Question 29. With the above interpretation, can we now interpret the wedge product as an oriented k-volume?

Question 30. Use the properties of the wedge product to compute the k-volumes of the following k-parallelotopes.

(1) $P(3\mathbf{e}_1 + 5\mathbf{e}_2, 4\mathbf{e}_2)$

(2) $P(\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, 3\mathbf{e}_1 + 6\mathbf{e}_2 + 3\mathbf{e}_3)$

(3) $P(3\mathbf{e}_1, 2\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)$

(4) $P(3\mathbf{e}_1, 2\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5)$

(1) The area of $P(3\mathbf{e}_1 + 5\mathbf{e}_2, 4\mathbf{e}_2)$ is the magnitude of $(3\mathbf{e}_1 + 5\mathbf{e}_2) \wedge (4\mathbf{e}_2)$.

$$\begin{aligned}(3\mathbf{e}_1 + 5\mathbf{e}_2) \wedge (4\mathbf{e}_2) &= (3\mathbf{e}_1) \wedge (4\mathbf{e}_2) + (5\mathbf{e}_2) \wedge (4\mathbf{e}_2) \\ &= 12(\mathbf{e}_1 \wedge \mathbf{e}_2) + 20(\mathbf{e}_2 \wedge \mathbf{e}_2) \\ &= 12(\mathbf{e}_1 \wedge \mathbf{e}_2)\end{aligned}$$

Since $\mathbf{e}_1 \wedge \mathbf{e}_2$ is the area of the unit square, $P(3\mathbf{e}_1 + 5\mathbf{e}_2, 4\mathbf{e}_2)$ has area 12.

Question 31. Let V be a vector space of the "magnitude and direction" sort. If $\mathbf{v}$ and $\mathbf{w}$ are in V and the angle between them is $\theta_{\mathbf{v},\mathbf{w}}$, what can you say about the magnitude of $\mathbf{v} \wedge \mathbf{w}$?

$\|\mathbf{v} \wedge \mathbf{w}\|$ is the area of the parallelogram $P(\mathbf{v}, \mathbf{w})$, so:

$$\|\mathbf{v} \wedge \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin(\theta_{\mathbf{v},\mathbf{w}}).$$

Remark 32. This may look familiar if you have encountered *cross products* before. The cross product is an operation on three dimensional vector spaces V of the magnitude and direction sort. While it has the magnitude described above, unlike the wedge product $\mathbf{v} \wedge \mathbf{w}$, the cross product $\mathbf{v} \times \mathbf{w}$ is also an element of V. It can be recovered from $\mathbf{v} \wedge \mathbf{w}$ by sending the area element $\mathbf{e}_i \wedge \mathbf{e}_{i+1}$ to the length element $\mathbf{e}_{i+2}$, where indices are taken modulo 3. (This is an instance of an operation called the *Hodge dual*.) So, we can easily recover the cross product from the wedge product – but the cross product is only defined in dimension three, and only for vector spaces of the magnitude and direction sort. And more importantly, while we *can* identify area elements with length elements in such a three dimensional vector space, I can't think of a single common use of the cross product where we *should* make this identification. In every instance that comes to mind, the concept represented by the cross product

(1) relates to an area not a length, and

(2) is fixed by negation of coordinates in V (like elements of $\bigwedge^2 V$, and in contrast to elements of V).

Remark 33. In Definition 26, we really described a richer algebraic structure than just a vector space. In fact, the wedge product can just as well be viewed as a map

$$\bigwedge^p V \times \bigwedge^q V \to \bigwedge^{p+q} V,$$

for any non-negative p and q. (Here $\bigwedge^0 V$ is interpretted as the scalars $\mathbb{R}$.) So, we have a larger structure that contains a vector space for each non-negative number, and a rule for multiplying elements of any of these vector spaces. What we have is called an *algebra*. This particular algebra is called the *exterior algebra of* V, and denoted $\bigwedge(V)$.

Question 34. If V is an n dimensional vector space, what is the dimension of $\bigwedge^k V$? If $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis for V, can you describe a basis for $\bigwedge^k V$?

Question 35. If V is an n dimensional vector space and $m > n$, what is the dimension of $\bigwedge^m V$? What is the dimension of the underlying vector space of the exterior algebra $\bigwedge(V)$?

Let's consider a couple examples from physics.

Example 36. *Torque* is a rotational analogue of force. While force induces a change in linear momentum ($\mathbf{F} = \frac{d\mathbf{p}}{dt}$), torque induces a change in angular momentum ($\tau = \frac{d\mathbf{L}}{dt}$). Torque about a pivot point p of an object results from applying a force $\mathbf{F}$ to another point q of that object. Let's denote the displacement vector $q - p$ by $\mathbf{r}$. Then $\tau = \mathbf{r} \wedge \mathbf{F}$.

Example 37. The Biot-Savart law for a point charge describes the *magnetic field* induced by a moving charged particle. If the particle has charge q and is moving with a velocity of $\mathbf{v}$ relative to the frame of measurement, the induced magnetic field at a displacement of $\mathbf{r}$ from the particle is

$$\frac{\mu_0}{4\pi} q \frac{\mathbf{v} \wedge \widehat{\mathbf{r}}}{\|\mathbf{r}\|^2}.$$

4 Surfaces in 3-space

We will now turn our attention to surfaces in 3-space. This simplest surface in 3-space is just a plane.

Question 38. If V is a three dimensional vector space and H is a plane in V, is H a two dimensional vector subspace of V? If not, what is it?

Answer. Not in general – only if H passes through the origin. Instead, it is a two dimensional affine space in V.

Since H is an affine space and not in general a vector space, we will often want to think of its points as elements of an affine space rather than a vector space. Since we can identify the points of an affine space and its associated vector space, we will sometimes view H as a subset of an affine space A whose associated vector space is V. In fact, I've been writing *3-space* as a cop-out to avoid specifying whether we are dealing with a three dimensional affine space or a three dimensional vector space. We can describe a plane as a subset of either, and there will just be a slight bit of translation work needed to go between these two descriptions. In what follows I will attempt to be clear about when we are choosing to view our plane as a subset of a vector space V and when we are choosing to view it as a subset of an affine space A.

There are multiple ways we can specify a plane H in a three dimensional vector space V or affine space A. For example, we can:

(1) Write an equation whose solution set is precisely H.

(2) List three non-collinear points in H. (This description fits more neatly in affine space setting. Why?)

(3) Give an area element parallel to H (so a wedge product in $\bigwedge^2 V$), and a displacement from the origin (any point in H).

(4) Give an area element parallel to H (say $\mathbf{u} \wedge \mathbf{v}$), and a signed volume $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ that will match for any $\mathbf{w}$ in H.

(5) Specify a vector $\mathbf{n}$ perpendicular to H and a displacement vector from the origin.

(6) Specify a vector $\mathbf{n}$ perpendicular to H and the dot product of $\mathbf{n}$ with elements of H.

Question 39. Let H be the plane containing the points $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. Can you write an equation for H? Can you describe H in terms of wedge products? Can you give a vector $\mathbf{n}$ perpendicular to H and a say what the dot product of $\mathbf{n}$ with points of H must be? Why isn't it 0?

In class, we described H in terms of a wedge product and then in terms of a dot product. Note that $(\mathbf{e}_2-\mathbf{e}_1)$ and $(\mathbf{e}_3-\mathbf{e}_1)$ are in the associated vector space of H, and $(\mathbf{e}_2-\mathbf{e}_1)\wedge(\mathbf{e}_3-\mathbf{e}_1)\wedge\mathbf{v}$ is the same for every $\mathbf{v} \in H$ – the associated parallelepipeds are all related by shears.

$$H = \{\mathbf{v} \in V \mid (\mathbf{e}_2 - \mathbf{e}_1) \wedge (\mathbf{e}_3 - \mathbf{e}_1) \wedge \mathbf{v} = (\mathbf{e}_2 - \mathbf{e}_1) \wedge (\mathbf{e}_3 - \mathbf{e}_1) \wedge \mathbf{e}_1\}$$

We simplified:

$$\begin{aligned}(\mathbf{e}_2 - \mathbf{e}_1) \wedge (\mathbf{e}_3 - \mathbf{e}_1) \wedge \mathbf{e}_1 &= (\mathbf{e}_2 - \mathbf{e}_1) \wedge \mathbf{e}_3 \wedge \mathbf{e}_1 \\ &= \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_1 \\ &= \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3,\end{aligned}$$

so

$$H = \{\mathbf{v} \in V \mid (\mathbf{e}_2 - \mathbf{e}_1) \wedge (\mathbf{e}_3 - \mathbf{e}_1) \wedge \mathbf{v} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3\}.$$

If we write $\mathbf{v}$ in coordinates as $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$, then

$$\begin{aligned} H &= \{\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 \mid (\mathbf{e}_2 - \mathbf{e}_1) \wedge (\mathbf{e}_3 - \mathbf{e}_1) \wedge (v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3\} \\ &= \cdots = \{\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 \mid (v_1 + v_2 + v_3)\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3\}. \end{aligned}$$

We concluded that $v_1 + v_2 + v_3 = 1$, and this simple equation defines H in coordinates.

Next, we noticed that the vector $\mathbf{n} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ is perpendicular to H, and that

$$H = \{\mathbf{v} \in V \mid \mathbf{n} \cdot \mathbf{v} = 1\}.$$

Finally, we identified these two descriptions by noting that the Hodge dual map that sends $\mathbf{v} \wedge \mathbf{w}$ to $\mathbf{v} \times \mathbf{w}$ recovers $\mathbf{n}$ from $(\mathbf{e}_2 - \mathbf{e}_1) \wedge (\mathbf{e}_3 - \mathbf{e}_1)$:

$$\begin{aligned} (\mathbf{e}_2 - \mathbf{e}_1) \wedge (\mathbf{e}_3 - \mathbf{e}_1) &= \mathbf{e}_2 \wedge \mathbf{e}_3 - \mathbf{e}_2 \wedge \mathbf{e}_1 - \mathbf{e}_1 \wedge \mathbf{e}_3 + \mathbf{e}_1 \wedge \mathbf{e}_1 \\ &= \mathbf{e}_2 \wedge \mathbf{e}_3 + \mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_1 \\ &= \mathbf{e}_2 \wedge \mathbf{e}_3 + \mathbf{e}_3 \wedge \mathbf{e}_1 + \mathbf{e}_1 \wedge \mathbf{e}_2 \\ &\mapsto \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \\ &= \mathbf{n}. \end{aligned}$$

In this problem it was easy to work out the line $\mathbf{n}$ must lie on, but when it isn't clear at a glance, the usual way to find $\mathbf{n}$ is with a cross product, in a way entirely analogous to the wedge product calculation we did above.

In the last question, you related some of the methods for specifying a plane in a particular example. Let's now relate them in a general setting. Suppose $\mathbf{n} = \langle a, b, c\rangle$ and $\mathbf{n} \cdot \langle x, y, z\rangle = d$ for any $\langle x, y, z\rangle \in H$. Then

$$H = \{\langle x, y, z\rangle \mid ax + by + cz = d\}. \tag{4}$$

Equation (4) is called the *general form of the equation of a plane.*

Question 40. If p, q, and r are three non-collinear points in H, can you describe an area element parallel to H?

Answer. One option – $(q - p) \wedge (r - p)$.

Question 41. Using your answer to Question 40 as your area element $\mathbf{A} \in \bigwedge^2 V$, can you fill in the blanks below?

$$H = \{s \in A \mid \mathbf{A} \wedge \underline{\qquad} = \mathbf{A} \wedge \underline{\qquad}\}$$

Answer. One option – $H = \{s \in A \mid \mathbf{A} \wedge (p - 0) = \mathbf{A} \wedge (s - 0)\}$.

As you saw in Equation (4), planes in 3-space are described by (affine) linear equations in our three coordinate variables – that is, degree 1 polynomial equations. The next surfaces we will discuss are described by quadratic polynomials in these coordinate variables – polynomials of degree 2. Such surfaces are called *quadric surfaces*.

Question 42. A sphere is the locus of points some fixed distance from a common (center) point. What is the equation for a sphere of radius r centered at the origin? What about a sphere of radius r centered at some point $p = (p_1, p_2, p_3)$?

Answer.

$$(x - p_1)^2 + (y - p_2)^2 + (z - p_3)^2 = r^2$$

Equations for spheres won't always be described so elegantly. So how do we recognize when an equation describes a sphere? We can expand the left side of the previous equation:

$$x^2 - 2xp_1 + {p_1}^2 + y^2 - 2yp_2 + {p_2}^2 + z^2 - 2zp_3 + {p_3}^2 = r^2$$

Collecting like terms, this has the form

$$x^2 + y^2 + z^2 + Ax + By + Cz + D = 0.$$

This has the same coefficient (1 in this case – but it just needs to be the same) for x^2, y^2, and z^2. There are no mixed terms – the coefficients of xy, xz, and yz are all 0. When we see an expression of the above form it *may* be a sphere. And we can figure out which sphere by completing the square:

Question 43. Does the following equation describe a sphere? If so, what are the center and radius?

$$x^2 + y^2 + z^2 - 2x - 4y + 8z + 17 = 0$$

Answer. Let's rearrange terms and complete the squares.

$$(x^2 - 2x) + (y^2 - 4y) + (z^2 + 8z) + 17 = 0$$

$$(x^2 - 2x + 1) + (y^2 - 4y + 4) + (z^2 + 8z + 16) + 17 - 1 - 4 - 16 = 0$$

$$(x - 1)^2 + (y - 2)^2 + (z + 4)^2 = 4$$

Yes, with center $(1, 2, -4)$ and radius 2.

Question 44. How could an equation of the form

$$x^2 + y^2 + z^2 + Ax + By + Cz + D = 0$$

fail to describe a sphere? What does it describe in this case?

Answer. Two situations can occur. When we rearrange terms to put our equation in the standard form of the equation of a sphere, the coefficient of r^2 may be:

- 0 – in this case the solution set is a single point.
- negative – in this case the solution set is empty.

Question 45. In 2-space, $x^2 + y^2 = r^2$ describes a circle of radius r centered at the origin. What about in 3-space?

Question 46. How would we describe a circle in 3-space? Try to describe a circle of radius r in the xy-plane centered at the origin.

Hint: *It takes more than one equation.*

Generically, each equation we add reduces the dimension of the solution set by one if there are common solutions.

Question 47. Why do I say *generically*? Can you come up with an example where the dimension decreases by more than one? Where the dimension does not change?

Question 48. Can we generalize our thoughts from Question 45? Was the circle important to that discussion? The process we are describing is called "extrusion", and the resulting surfaces are called cylindrical surfaces.

Example 49 (Example 2.55.c. of the Textbook)**.** Sketch the surface

$$\{(x, y, z) \mid y = \sin(x)\}.$$

Is this a quadric surface?

Caution: You may need to do a change of variables to recognize a cylindrical surface. The extrusion axis may not be a coordinate axis in your original coordinate system.

Example 50 (Example 2.55.b. of the Textbook)**.** Sketch the surface

$$\left\{(x, y, z) \;\middle|\; z = 2x^2 - y\right\}.$$

What is the extrusion axis in this case? Can you make a change of variables so that the extrusion axis is one of the coordinate axes of your new coordinate system?

Now let's look at some more possibilities for quadric surfaces. Since quadric surfaces are the solution sets in 3-space of quadratic polynomials in three variables, in general we are looking at surfaces cut out by equations of the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Jz + K = 0,$$

with at least one coefficient of a degree two term non-zero.

We can get an idea of what these surfaces look like by intersecting with planes. Generally, we take slices parallel to the coordinate axes. The resulting intersections are called traces.

Question 51 (Example 2.56 of the Textbook)**.** Can you sketch the surface cut out by the equation

$$\frac{1}{4}x^2 + \frac{1}{9}y^2 + \frac{1}{25}z^2 = 1?$$

This is what we call an *ellipsoid*. Let's see how ellipsoids can be described more generally now.

Question 52. Can you sketch the surface cut out by the equation

$$\frac{1}{a^2}(x - p_1)^2 + \frac{1}{b^2}(y - p_2)^2 + \frac{1}{c^2}(z - p_3)^2 = 1?$$

You might notice that an ellipsoid has three distinguished axes – called the *principal axes* of the ellipsoid – which are perpendicular to one another. In the previous example, these axes we parallel to the coordinate axes.

Question 53. Can you give an equation for an ellipsoid centered at the origin and having principal axes parallel to $\mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{e}_1 - \mathbf{e}_2$, and $\mathbf{e}_3$?

Answer. First, let's find unit vectors in the stated directions.

$$\mathbf{e}_1' = \frac{\mathbf{e}_1 + \mathbf{e}_2}{\|\mathbf{e}_1 + \mathbf{e}_2\|} = \frac{\mathbf{e}_1 + \mathbf{e}_2}{\sqrt{2}}, \qquad \mathbf{e}_2' = \frac{\mathbf{e}_1 - \mathbf{e}_2}{\|\mathbf{e}_1 - \mathbf{e}_2\|} = \frac{\mathbf{e}_1 - \mathbf{e}_2}{\sqrt{2}}, \qquad \mathbf{e}_3' = \mathbf{e}_3.$$

Going the other way, we can rewrite the original basis vectors in terms of the new basis vectors:

$$\frac{\mathbf{e}_1' + \mathbf{e}_2'}{\sqrt{2}} = \frac{2\mathbf{e}_1}{(\sqrt{2})^2} = \mathbf{e}_1, \qquad \frac{\mathbf{e}_1' - \mathbf{e}_2'}{\sqrt{2}} = \frac{2\mathbf{e}_2}{(\sqrt{2})^2} = \mathbf{e}_2, \qquad \mathbf{e}_3 = \mathbf{e}_3'.$$

Now describe an arbitrary point in both coordinate systems:

$$\begin{aligned} x'\mathbf{e}_1' + y'\mathbf{e}_2' + z'\mathbf{e}_3' &= x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 \\ &= x\frac{\mathbf{e}_1' + \mathbf{e}_2'}{\sqrt{2}} + y\frac{\mathbf{e}_1' - \mathbf{e}_2'}{\sqrt{2}} + z\mathbf{e}_3' \\ &= \frac{x+y}{\sqrt{2}}\mathbf{e}_1' + \frac{x-y}{\sqrt{2}}\mathbf{e}_2' + z\mathbf{e}_3'. \end{aligned}$$

So, we can relate the coefficients by

$$x' = \frac{x+y}{\sqrt{2}}, \qquad y' = \frac{x-y}{\sqrt{2}}, \qquad z' = z.$$

Now let's suppose we have the ellipsoid

$$\frac{1}{a^2}(x')^2 + \frac{1}{b^2}(y')^2 + \frac{1}{c^2}(z')^2 = 1.$$

In terms of the original coordinate system, this reads

$$
\begin{aligned}
1 &= \frac{1}{a^2}\frac{(x+y)^2}{2} + \frac{1}{b^2}\frac{(x-y)^2}{2} + \frac{1}{c^2}z^2 \\
&= \frac{1}{a^2}\frac{(x+y)^2}{2} + \frac{1}{b^2}\frac{(x-y)^2}{2} + \frac{1}{c^2}z^2 \\
&= \frac{1}{a^2}\frac{x^2+2xy+y^2}{2} + \frac{1}{b^2}\frac{x^2-2xy+y^2}{2} + \frac{1}{c^2}z^2 \\
&= \frac{1}{2(a^2+b^2)}x^2 + \frac{1}{2(a^2+b^2)}y^2 + \frac{1}{c^2}z^2 + \left(\frac{1}{a^2} - \frac{1}{b^2}\right)xy
\end{aligned}
$$

Notice that this has a *cross term* – the coefficient of xy is non-zero if $a^2 \neq b^2$.

Question 54. Can you sketch the surface cut out by the equation

$$\frac{1}{a^2}x^2 + \frac{1}{b^2}y^2 - \frac{1}{c^2}z^2 = 1?$$

Do we have a name for the sort of curves we find as traces parallel to the xy-plane? yz-plane? xz-plane?

This is what we call a one-sheeted hyperboloid.

Question 55. Can you sketch the surface cut out by the equation

$$-\frac{1}{a^2}x^2 - \frac{1}{b^2}y^2 + \frac{1}{c^2}z^2 = 1?$$

Do we have a name for the sort of curves we find as traces parallel to the xy-plane? yz-plane? xz-plane?

This is what we call a two-sheeted hyperboloid.

Question 56. Can you give an equation for the cone whose symmetry axis is the z-axis and angle (between the z-axis and surface) is $\frac{\pi}{4}$? Can you describe the traces parallel to the xy-plane? yz-plane? xz-plane?

There is a very handy chart of common quadric surfaces on pages 201 and 202 of the text.