Littlewood-Richardson coefficients from Mirror Symmetry

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Motivation

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How do we compute the **Littlewood-Richardson coefficients** $c_{\alpha,\beta}^\gamma$?

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- Review of Knutson-Tao hive cone

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- Representation theory set-up: Which space should we study?
- Log Calabi-Yau mirror symmetry background and solution

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Advantages of Mirror Symmetry Approach

- Lots of cones, intrinsic "cone"
- Canonical Bases
- Very general construction

Knutson-Tao Hive Cone ([KT98])

Alternative Formulation of Littlewood-Richardson coefficients

$$V_\alpha \otimes V_\beta = \bigoplus_\gamma V_\gamma^{\oplus c_{\alpha,\beta}^\gamma}$$

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$$V_\alpha \otimes V_\beta \otimes V_\delta = \bigoplus_\gamma V_\gamma^{\oplus c_{\alpha,\beta}^{\gamma}} \otimes V_\delta$$

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Alternative Formulation of Littlewood-Richardson coefficients

$$\left(V_\alpha \otimes V_\beta \otimes V_\delta\right)^{\mathrm{GL}_n} = \left(\bigoplus_\gamma V_\gamma^{\oplus c_{\alpha,\beta}^\gamma} \otimes V_\delta\right)^{\mathrm{GL}_n}$$

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$$\left(V_\alpha \otimes V_\beta \otimes V_\delta\right)^{\mathrm{GL}_n} = \mathrm{id}_{\mathbb{C}}^{\oplus c_{\alpha,\beta}^{-w_0(\delta)}}$$

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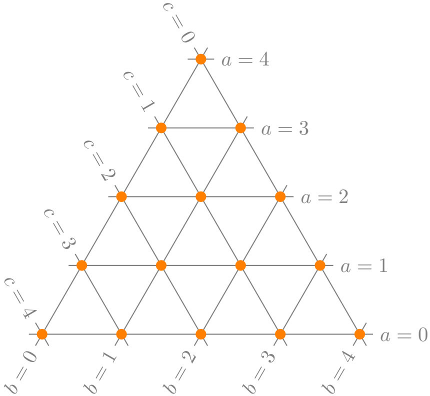
Given three irreducible GL_n representations V_α, V_β, V_γ, what is the dimension of $(V_\alpha \otimes V_\beta \otimes V_\gamma)^{\mathrm{GL}_n}$?

Knutson-Tao Hive Cone ([KT98])

- Start with $\mathcal{H} = \left\{ (a,b,c) \in (\mathbb{Z}_{\geq 0})^3 \;\middle|\; a+b+c=n \right\}$. Take real labelings $\mathbb{R}^{\mathcal{H}}$.



$\mathcal{H}$ for $n=4$

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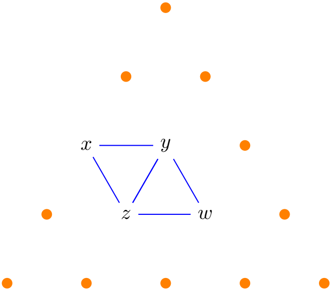
- Start with $\mathcal{H} = \left\{ (a,b,c) \in (\mathbb{Z}_{\geq 0})^3 \;\middle|\; a+b+c=n \right\}$. Take real labelings $\mathbb{R}^{\mathcal{H}}$.
- Impose rhombus inequalities.



$\rightsquigarrow$ $y + z \geq x + w$

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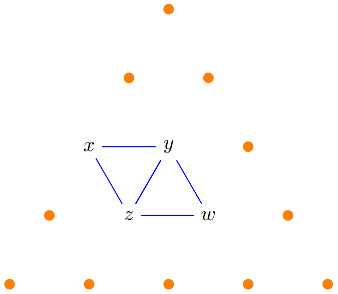
- Impose rhombus inequalities.
- Take quotient by linear subspace spanned by $\mathbf{1}_{\mathcal{H}}$.



Result is the **Knutson-Tao hive cone**.

Its points are called **hives**.

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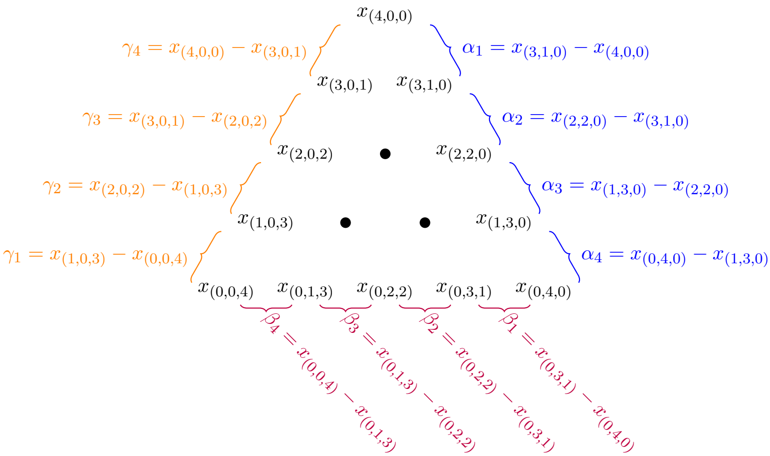
To compute $\dim\left(V_\alpha \otimes V_\beta \otimes V_\gamma\right)^{\mathrm{GL}_n}$:

- Fill the border as shown.



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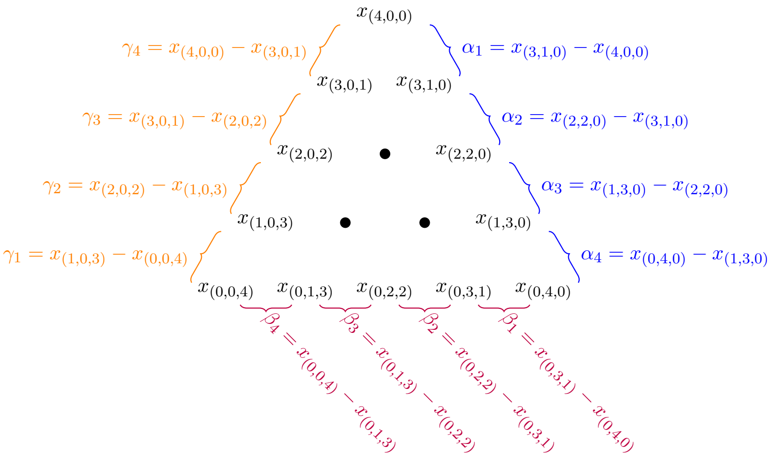
To compute $\dim\left(V_\alpha \otimes V_\beta \otimes V_\gamma\right)^{\mathrm{GL}_n}$:

- Fill the border as shown.
- Count integral hives with this border.



The Set-up

Petr-Weyl Theorem

Let G be a reductive group. As $G \times G$ bimodules

$$\mathcal{O}(G) = \bigoplus_{\lambda} V_\lambda \otimes V_\lambda^*$$

where the sum is over isomorphism classes of irreducible rational representations of G.

The Set-up

Consequence for $G = \mathrm{GL}_n$

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$$\mathcal{O}(\mathrm{GL}_n)^{1 \times U} = \Big(\bigoplus_{\lambda} V_\lambda \otimes V_\lambda^*\Big)^{1 \times U}$$

where U consists of upper triangular matrices with all diagonal entries 1.

The Set-up

Consequence for $G = \mathrm{GL}_n$

$$\mathcal{O}(\mathrm{GL}_n / U) = \bigoplus_{\lambda} V_\lambda \otimes \mathbb{C} \cdot u_\lambda$$

where u_λ is the highest weight vector of V_λ^*.

The Set-up

Consequence for $G = \mathrm{GL}_n$

$$\mathcal{O}(\mathrm{GL}_n / U) = \bigoplus_{\lambda} V_\lambda$$

This is a weight space decomposition for right action of maximal torus, with V_λ the $-w_0(\lambda)$ weight space.

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$\mathrm{Conf}_3(\mathrm{GL}_n /U) := \mathrm{GL}_n \backslash (\mathrm{GL}_n /U)^{\times 3}$ defined and studied in [FG06], [GS15].

Cluster Varieties: Context and Definition ([GHK15])

Definition (*Log Calabi-Yau variety*)

A smooth complex variety U with a unique volume form Ω having at worst a simple pole along any divisor in *any* compactification of U

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Algebraic torus $T = (\mathbb{C}^*)^n$, $\Omega = \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$
Fact: If (Y, D) is any toric variety with its toric boundary divisor, then Ω has a simple pole along each component of D.

Cluster Varieties: Context and Definition ([GHK15])

Example (Carefully glued tori)

$$U = \bigcup_i T_i / \sim$$

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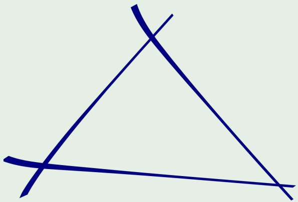
- $(\overline{Y}, \overline{D})$ toric variety



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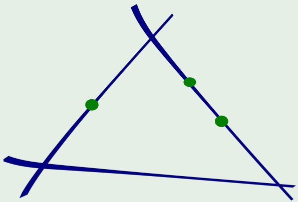

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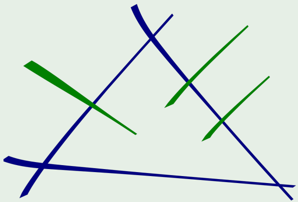

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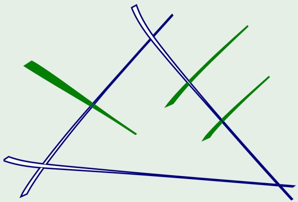

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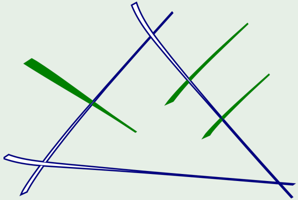

Two ways of describing a **Cluster Variety**

Cluster Structure of $\mathrm{Conf}_3(\widetilde{\mathcal{F}\ell})$ ([FG06], [GS15])

Decorated flags

- Let a **decorated flag** be a complete flag $X_\bullet = (X_1 \subset \cdots \subset X_n)$, together with a non-zero vector $x_i \in X_i/X_{i-1}$ in each successive quotient.

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- $(X_\bullet, Y_\bullet)$ is in **generic position** if X_i and Y_{n-i} intersect transversely for all i.
- $\mathrm{Conf}_3^\times(\widetilde{\mathcal{F}\ell}) \subset \mathrm{Conf}_3(\widetilde{\mathcal{F}\ell})$ is the locus where triples of decorated flags $(X, Y, Z) := ((X_\bullet, x_\bullet), (Y_\bullet, y_\bullet), (Z_\bullet, z_\bullet))$ are in generic configuration, meaning each pair of flags is in generic position.

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- Define $A_{(a,b,c)} : (X, Y, Z) \mapsto \omega(x_1, \ldots, x_a, y_1, \ldots, y_b, z_1, \ldots, z_c)$.

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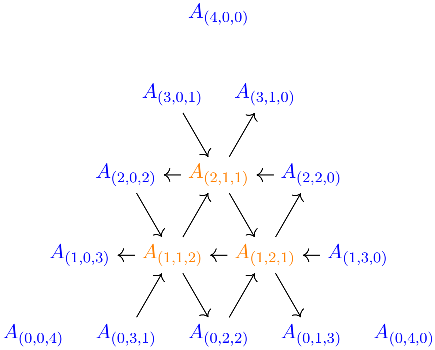

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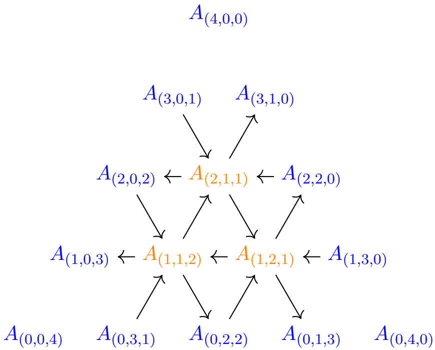

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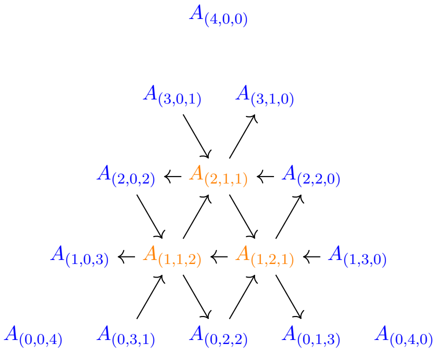

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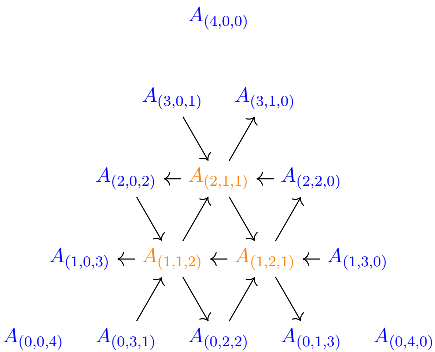

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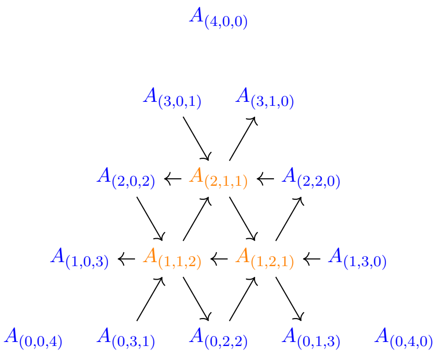

- Initial cluster and quiver for a torus in $\mathrm{SL}_n \backslash \widetilde{\mathcal{F}\ell}^{\times 3}$.
  - Note that $A_{(a,b,c)}$ respects the quotient by SL_n.
- $\displaystyle\prod_{a+b+c=n} A_{(a,b,c)}^{r_{(a,b,c)}}$ is fixed by GL_n if and only if $\displaystyle\sum_{a+b+c=n} r_{(a,b,c)} = 0$.

Cox type constuction

$$T^{\vee}_{\mathrm{Pic}(\mathrm{Conf}_3(\mathcal{F}\ell))} = H^{\times 3}$$

$$\begin{array}{ccc}
\mathrm{Conf}_3^{\times}(\widetilde{\mathcal{F}\ell}) & \hookrightarrow & \mathrm{Conf}_3(\widetilde{\mathcal{F}\ell}) \circlearrowleft \\
\downarrow & & \downarrow \\
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Cox type constuction

Like $(\mathbb{C}^*)^{\Sigma(1)}$

Like $\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$

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Like defining torus T

Like X_Σ

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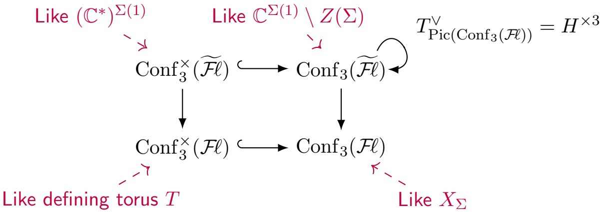

Strategy

- Equip $\mathcal{O}(\mathrm{Conf}_3^\times(\widetilde{\mathcal{F}\ell}))$ with a canonical basis $\mathbf{B}^\times$.

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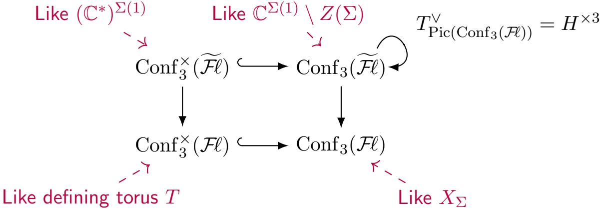

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- Describe $\mathbf{B} := \mathbf{B}^\times \cap \mathcal{O}(\mathrm{Conf}_{\mathbf{3}}(\widetilde{\mathcal{F}\ell}))$ and show it is a basis for $\mathcal{O}(\mathrm{Conf}_3(\widetilde{\mathcal{F}\ell})) = \bigoplus_{\alpha,\beta,\gamma} (V_\alpha \otimes V_\beta \otimes V_\gamma)^{\mathrm{GL}_n}$.

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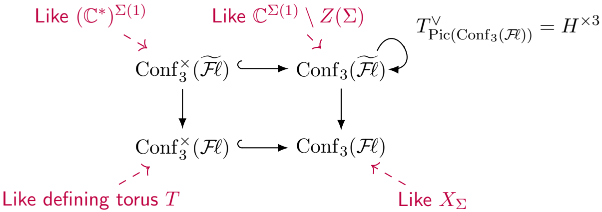

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- Show that elements of $\mathbf{B}$ are eigenfunctions of $H^{\times 3}$ action, so we get a basis for each summand.

Tropicalization of a log Calabi-Yau ([GHKK18])

Definition

Let (U, Ω) be log CY. A **divisorial discrete valuation** (ddv) $\nu : \mathbb{C}(U) \setminus \{0\} \to \mathbb{Z}$ is a discrete valuation of the form $\nu = \mathrm{ord}_D(\,\cdot\,)$ where D is (a positive multiple of) an irreducible effective divisor in a variety birational to U.

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Example

If $U = T_N$, $U^{\mathrm{trop}}(\mathbb{Z}) = N$.

Tropicalization of a log Calabi-Yau ([GHKK18])

Definition

Let (U, Ω) be log CY. A **divisorial discrete valuation** (ddv) $\nu : \mathbb{C}(U) \setminus \{0\} \to \mathbb{Z}$ is a discrete valuation of the form $\nu = \mathrm{ord}_D(\,\cdot\,)$ where D is (a positive multiple of) an irreducible effective divisor in a variety birational to U. The **integral tropicalization of** U is $U^{\mathrm{trop}}(\mathbb{Z}) := \{\nu \text{ ddv} : \nu(\Omega) < 0\} \cup \{0\}$.

Example

If $U = T_N$, $U^{\mathrm{trop}}(\mathbb{Z}) = N$. Recall that toric divisors are indexed by cocharacters.

Tropicalization of a log Calabi-Yau variety ([GHKK18])

Remark

- We can extend scalars from $\mathbb{Z}_{>0}$ to $\mathbb{R}_{>0}$ in the definition of $U^{\mathrm{trop}}(\mathbb{Z})$ to obtain $U^{\mathrm{trop}}(\mathbb{R})$ – the **real tropicalization of** U.

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- $U^{\mathrm{trop}}(\mathbb{R})$ has a natural piecewise linear structure.
- When $U = T_N$, $U^{\mathrm{trop}}(\mathbb{R}) = N_{\mathbb{R}}$ is actually linear.

Dual Basis Conjectures

Conjecture of Gross-Hacking-Keel

Let U be an affine log Calabi-Yau with *maximal boundary*

Dual Basis Conjectures

Conjecture of Gross-Hacking-Keel

Let U be an affine log Calabi-Yau with *maximal boundary* – this means it has a minimal model (Y, D) where D has a 0-stratum.

Dual Basis Conjectures

Conjecture of Gross-Hacking-Keel

Let U be an affine log Calabi-Yau with *maximal boundary*. Then the mirror $U^\vee$ is again an affine log Calabi-Yau with maximal boundary. The integral tropical points of U parametrize a basis of ϑ-functions on $U^\vee$, with multiplication given explicitly in terms of broken line counts.

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Cluster case

If U is a cluster variety, this is (a version of) the dual basis conjecture of Fock and Goncharov. ([FG09]) Here, U and $U^\vee$ are built out of dual tori.

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If U is a cluster variety, this is (a version of) the dual basis conjecture of Fock and Goncharov. ([FG09]) Here, U and $U^\vee$ are built out of dual tori. Conditions implying the conjecture holds for U are given in [GHKK18].

Dual Basis Conjectures

Theorem (M.)

$(\mathrm{Conf}_3^\times(\widetilde{\mathcal{F}\ell})^\vee)^{\mathrm{trop}}(\mathbb{Z})$ *parametrizes a canonical basis for* $\mathcal{O}(\mathrm{Conf}_3^\times(\widetilde{\mathcal{F}\ell}))$.

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- In the toric analogy, this parametrization is like saying the cocharacters of $T^\vee$ parametrize a canonical basis for $\mathcal{O}(T)$.

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- This is the basis $\mathbf{B}^\times$.
- The next step is to cut it down to $\mathbf{B}$.

Landau-Ginzburg Potential and the basis $\mathbb{B}$

- Let $D_{(a,b,c)} \subset \mathrm{Conf}_3(\widetilde{\mathcal{F}\ell})$ be the vanishing locus of $A_{(a,b,c)}$.

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- $\nu_{(a,b,c)}$ defines a ϑ-function $\vartheta_{(a,b,c)}$ on $\mathrm{Conf}_3^\times(\widetilde{\mathcal{F}\ell})^\vee$.
- The Landau-Ginzburg mirror to $\left(\mathrm{Conf}_3(\widetilde{\mathcal{F}\ell}), D\right)$ is

  $$W = \sum_{\text{Exactly } 1 \text{ of } a,b,c \text{ is } 0} \vartheta_{(a,b,c)} : \mathrm{Conf}_3^\times(\widetilde{\mathcal{F}\ell})^\vee \to \mathbb{C}.$$

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$$\begin{aligned}\langle\,\cdot\,,\,\cdot\,\rangle : \mathrm{Conf}_3^\times(\widetilde{\mathcal{F}\ell})^{\mathrm{trop}}(\mathbb{Z}) \times (\mathrm{Conf}_3^\times(\widetilde{\mathcal{F}\ell})^\vee)^{\mathrm{trop}}(\mathbb{Z}) &\to \mathbb{Z}\\ (\,\nu\,,\,p\,) &\mapsto \nu(\vartheta_p)\end{aligned}$$

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- Swapping $\mathrm{Conf}_3^\times(\widetilde{\mathcal{F}\ell})$ and $\mathrm{Conf}_3^\times(\widetilde{\mathcal{F}\ell})^\vee$ gives pairing $\langle \cdot \, , \, \cdot \rangle^\vee$.

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- Conjecturally $\langle \nu, p\rangle = \langle \nu, p\rangle^\vee$ in general. Equality is known if ϑ_ν or ϑ_p restricts to a character on some cluster torus.

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Theorem (M.)

Every summand $\vartheta_{(a,b,c)}$ of W restricts to a character on some cluster torus. As a result, $p(\vartheta_{(a,b,c)}) = \langle \nu_{(a,b,c)}, p\rangle^\vee = \langle \nu_{(a,b,c)}, p\rangle = \nu_{(a,b,c)}(\vartheta_p)$– the order of vanishing of ϑ_p along $D_{(a,b,c)}$.

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Every summand $\vartheta_{(a,b,c)}$ *of* W *restricts to a character on some cluster torus. As a result,* $p(\vartheta_{(a,b,c)}) = \langle \nu_{(a,b,c)}, p \rangle^{\vee} = \langle \nu_{(a,b,c)}, p \rangle = \nu_{(a,b,c)}(\vartheta_p)$*– the order of vanishing of* ϑ_p *along* $D_{(a,b,c)}$*.*

Corollary

Define $\vartheta_{\nu}^{\mathrm{trop}}(p) = p(\vartheta_{\nu})$ *and*

$$W^{\mathrm{trop}} := \min \vartheta_{(a,b,c)}^{\mathrm{trop}} : (\mathrm{Conf}_3^{\times}(\widetilde{\mathcal{F}\ell})^{\vee})^{\mathrm{trop}}(\mathbb{Z}) \to \mathbb{Z}.$$

Then $\Xi(\mathbb{Z}) := \left\{ p \in (\mathrm{Conf}_3^{\times}(\widetilde{\mathcal{F}\ell})^{\vee})^{\mathrm{trop}}(\mathbb{Z}) \;\middle|\; W^{\mathrm{trop}}(p) \geq 0 \right\}$ *parametrizes the elements of* $\mathbf{B}^{\times}$ *that extend to* $\mathrm{Conf}_3(\widetilde{\mathcal{F}\ell})$*.*

Landau-Ginzburg Potential and the basis $\mathbb{B}$

Remark

Doesn't quite establish that $\Xi(\mathbb{Z})$ parametrizes a basis for $\mathcal{O}(\mathrm{Conf}_3(\widetilde{\mathcal{F}\ell}))$– it would fail, *e.g.* if ϑ_p and ϑ_q have poles on D, but the poles cancel in $\vartheta_p + \vartheta_q$.

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Theorem (Gross-Hacking-Keel-Kontsevich)

Suppose ϑ_ν restricts to a character on some cluster torus. If $\nu(\sum_p c_p\vartheta_p) \geq 0$, then $\nu(\vartheta_p) \geq 0$ for all p with $c_p \neq 0$.

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Corollary

$\Xi(\mathbb{Z})$ parametrizes a basis $\mathbf{B}$ for $\mathcal{O}(\mathrm{Conf}_3(\widetilde{\mathcal{F}\ell}))$.

ϑ-functions are eigenfunctions of $H^{\times 3}$ action

- $\mathbf{B}$ only depends on inclusion $\mathrm{Conf}_3^{\times}(\widetilde{\mathcal{F}\ell}) \subset \mathrm{Conf}_3(\widetilde{\mathcal{F}\ell})$.

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Moreover:

Theorem (M.)

There is a map $\mathrm{Conf}_3^{\times}(\widetilde{\mathcal{F}\ell})^{\vee} \to (H^{\times 3})^{\vee}$ *whose tropicalization* w *satisfies:*

$$w : \left(\mathrm{Conf}_3^{\times}(\widetilde{\mathcal{F}\ell})^{\vee}\right)^{\mathrm{trop}}(\mathbb{Z}) \to \mathbb{Z}$$

$$q \mapsto H^{\times 3} - \textit{weight of } \vartheta_q.$$

Then $P_{\alpha,\beta,\gamma}(\mathbb{Z}) := w^{-1}\left(-w_0(\alpha), -w_0(\beta) - w_0(\gamma)\right) \cap \Xi(\mathbb{Z})$ *parametrizes a basis for* $(V_\alpha \otimes V_\beta \otimes V_\gamma)^{\mathrm{GL}_n}$.

Explicit description of Ξ and $P_{\alpha,\beta,\gamma}$

- Ξ and $P_{\alpha,\beta,\gamma}$ are subsets of the piecewise linear manifold $(\mathrm{Conf}_3^\times(\widetilde{\mathcal{F}\ell})^\vee)^{\mathrm{trop}}(\mathbb{R})$.

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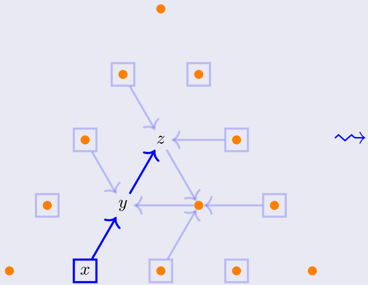
Ξ for initial cluster torus



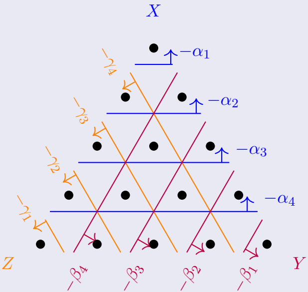
$$x \geq 0$$

$$x + y \geq 0$$

$$x + y + z \geq 0$$

Explicit description of Ξ and $P_{\alpha,\beta,\gamma}$

$P_{\alpha,\beta,\gamma}$ for initial cluster torus



$\rightsquigarrow$ Entries on indicated side of line sum to the given value

Connection to [GS15] and the Knutson-Tao Hive Cone

Goncharov-Shen Potential

- Goncharov and Shen define a potential on $\mathrm{Conf}_3^\times(\widetilde{\mathcal{F}\ell})$ (rather than its mirror).

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- For $((U_1, \chi_1), (U_2, \chi_2), (U_3, \chi_3)) \in \mathrm{Conf}_3^\times(\widetilde{\mathcal{F}\ell})$, there is a unique $u_{jk} \in U_i$ conjugating U_j to U_k.

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- Results in natural function on $\mathrm{Conf}_3^\times(\widetilde{\mathcal{F}\ell})$:

$$\mathcal{W}_{\mathsf{GS}} : \mathrm{Conf}_3^\times(\widetilde{\mathcal{F}\ell}) \to \mathbb{C}$$
$$((U_1, \chi_1), (U_2, \chi_2), (U_3, \chi_3)) \mapsto \chi_1(u_{23}) + \chi_2(u_{31}) + \chi_3(u_{12})$$

Connection to [GS15] and the Knutson-Tao Hive Cone

Theorem (Goncharov-Shen)

For the initial cluster torus

$$\left\{ \nu \in \mathrm{Conf}_3^\times(\widetilde{\mathcal{F}\ell})^{\mathrm{trop}}(\mathbb{R}) \;\middle|\; \mathcal{W}_{GS}^{\mathrm{trop}}(\nu) \geq 0 \right\}$$

is precisely the Knutson-Tao hive cone.

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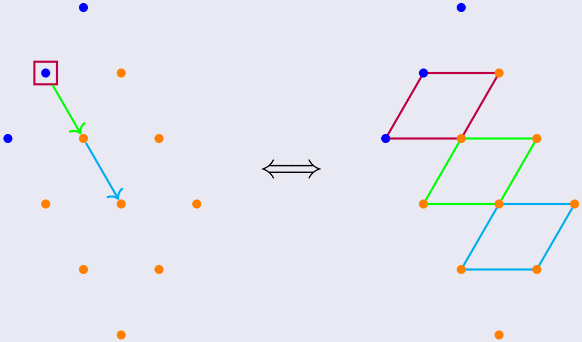
Theorem (M.)

There is an isomorphism $p : \mathrm{Conf}_3^\times(\widetilde{\mathcal{F}\ell}) \to \mathrm{Conf}_3^\times(\widetilde{\mathcal{F}\ell})^\vee$ *with* $p^*(W) = \mathcal{W}_{GS}$*. For the initial cluster tori, this identifies* Ξ *with the Knutson-Tao hive cone.*

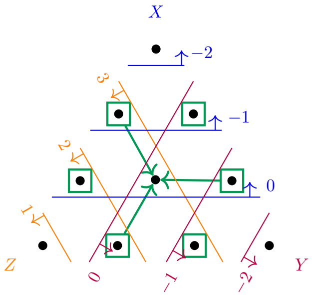
Connection to [GS15] and the Knutson-Tao Hive Cone

Identifying the cones pictorially

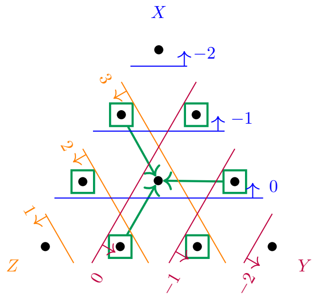


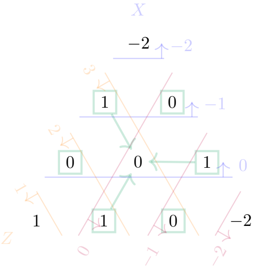
Example

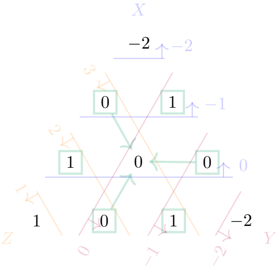
Computing $c^{(3,2,1)}_{(2,1,0),(2,1,0)} = \dim\left(V_{(2,1,0)} \otimes V_{(2,1,0)} \otimes V_{(-1,-2,-3)}\right)^{\mathrm{GL}_3}$



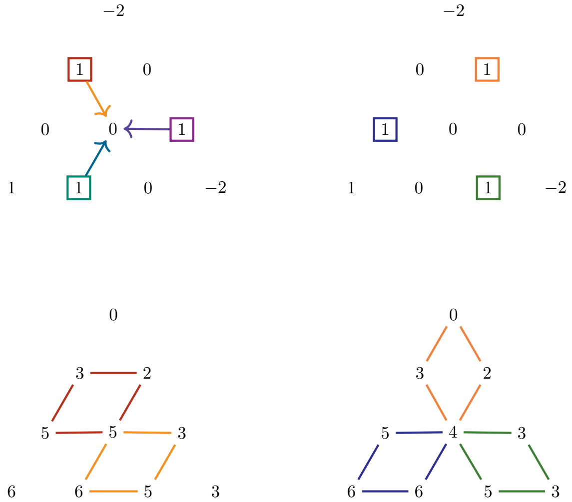
Example ($c^{(3,2,1)}_{(2,1,0),(2,1,0)} = 2$)



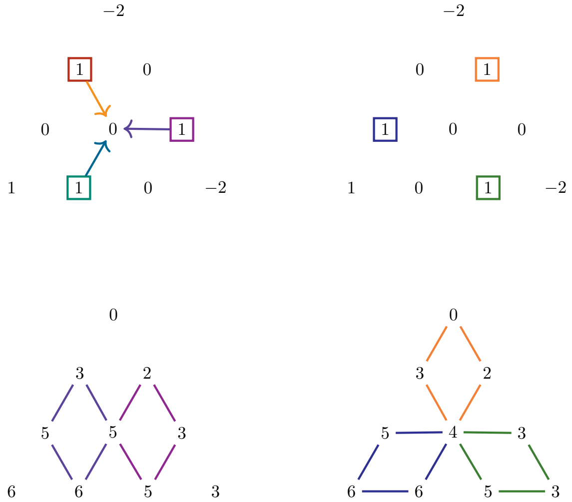




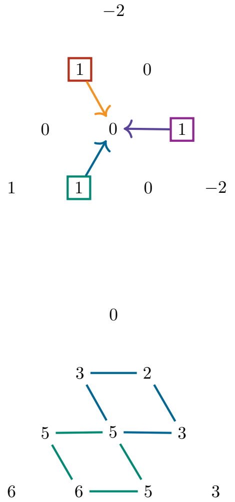
Example ($c^{(3,2,1)}_{(2,1,0),(2,1,0)} = 2$)



Example ($c^{(3,2,1)}_{(2,1,0),(2,1,0)} = 2$)



Example ($c^{(3,2,1)}_{(2,1,0),(2,1,0)} = 2$)



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