Grassmannians, plabic graphs, and mirror symmetry for cluster varieties

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Ongoing joint work with Lara Bossinger, Mandy Cheung, and Alfredo Nájera Chávez

Cluster varieties (skew symmetric type, [FG09], [GHK15a])

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$$\Gamma := \begin{cases} \text{Lattice } N \text{ with skew-form } \{\cdot,\cdot\}: N \times N \to \mathbb{Z} \\ \text{Saturated sublattice } N_{\mathrm{uf}} \subset N \\ \text{Index set } I \text{ with } |I| = \mathrm{rank}(N) \text{ and subset } I_{\mathrm{uf}} \text{ with } |I_{\mathrm{uf}}| = \mathrm{rank}(N_{\mathrm{uf}}) \end{cases}$$

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Let $M := \operatorname{Hom}(N, \mathbb{Z})$.

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$$T_{M;\mathbf{s}} := \operatorname{Spec}(\mathbb{C}[N]) \qquad T_{N;\mathbf{s}} := \operatorname{Spec}(\mathbb{C}[M])$$

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Mutation

For $k \in I_{\mathrm{uf}}$:

- $\mu_k(\mathbf{s}) := (e'_i : i \in I)$ where $e'_i := \begin{cases} e_i + [\{e_i, e_k\}]_+ e_k & \text{for } i \neq k \\ -e_k & \text{for } i = k \end{cases}$

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- $T_{M;\mathbf{s}} \dashrightarrow T_{M;\mu_k(\mathbf{s})}$ defined in terms of pullback of functions by $\mu_k^*(z^n) = z^n \left(1 + z^{e_k}\right)^{-\{n, e_k\}}$

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Two flavors of cluster varieties

$$\mathcal{X}_{\Gamma,[\mathbf{s}]} := \bigcup_{\mathbf{s}' \in [\mathbf{s}]} T_{M;\mathbf{s}'} / \sim \qquad \mathcal{A}_{\Gamma,[\mathbf{s}]} := \bigcup_{\mathbf{s}' \in [\mathbf{s}]} T_{N;\mathbf{s}'} / \sim$$

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Definition

Denote the projection $M \to M/N_{\mathrm{uf}}^{\perp}$ by π. A **cluster ensemble lattice map** is a map $p^* : N \to M$ such that

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Definition

A cluster ensemble lattice map p^* defines a **cluster ensemble map** $p : \mathcal{A} \to \mathcal{X}$ in terms of pullback of functions.

$$p^*(z^n) = z^{p^*(n)}$$

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Fact/Definition

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Algebraic torus $T = (\mathbb{C}^*)^n$, $\Omega = \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$.

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If (Y, D) is any toric variety with toric boundary divisor, Ω has a simple pole along each component of D.

Definition

Let $(\mathcal{V}, \Omega)$ be log Calabi-Yau scheme. A **divisorial discrete valuation** (ddv) $\nu : \mathbb{C}(\mathcal{V}) \setminus 0 \to \mathbb{Z}$ is a discrete valuation of the form $\nu = \mathrm{ord}_D(\,\cdot\,)$ where D is (a positive multiple of) an irreducible effective divisor in a variety birational to $\mathcal{V}$.

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Recall that toric divisors are indexed by cocharacters.

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This means there is some compactification (Y, D) of $\mathcal{V}$ such that Ω has a pole along all divisorial components of D and D has a 0-stratum.

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Let $\mathcal{V}$ be an affine log Calabi-Yau with maximal boundary. Then

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This is known as the ϑ-basis, with elements written as ϑ_ν for $\nu \in \mathcal{V}^{\mathrm{trop}}(\mathbb{Z})$.

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- For such cluster varieties $\mathcal{V}$, it is said that **the full Fock-Goncharov conjecture holds for $\mathcal{V}$**.
- Marsh-Scott show that the full Fock-Goncharov conjecture holds for the cluster varieties associated to Grassmannians ([MS16]).

Induced maps of mirrors

Definition

Given a rational map $f : \mathcal{U} \dashrightarrow \mathcal{V}$ of log Calabi-Yaus with $f^*(\Omega_{\mathcal{V}}) = \Omega_{\mathcal{U}}$, the tropicalization of f is

$$\begin{aligned} f^{\mathrm{trop}} : \mathcal{U}^{\mathrm{trop}}(\mathbb{Z}) &\to \mathcal{V}^{\mathrm{trop}}(\mathbb{Z}) \\ v &\mapsto v \circ f^*. \end{aligned}$$

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Assuming the conjecture holds...

Let $f : \mathcal{U} \to \mathcal{V}$ be a map of affine log Calabi-Yaus with maximal boundary satisfying:

If $R\left(\vartheta_v : v \in \mathcal{U}^{\mathrm{trop}}(\mathbb{Z})\right)$ *is a relation in* $A_{\mathcal{U}}$, *then* $R\left(\vartheta_{f^{\mathrm{trop}}(v)} : v \in \mathcal{U}^{\mathrm{trop}}(\mathbb{Z})\right)$ *is a relation in* $A_{\mathcal{V}}$.

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Then $f^{\text{trop}} : \mathcal{U}^{\text{trop}}(\mathbb{Z}) \to \mathcal{V}^{\text{trop}}(\mathbb{Z})$ determines a map of algebras $A_{\mathcal{U}} \to A_{\mathcal{V}}$, and so a map of schemes $f^{\vee} : \mathcal{V}^{\vee} \to \mathcal{U}^{\vee}$.

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Proposition (Bossinger, Cheung, M, Nájera Chávez)

Assume the full Fock-Goncharov conjecture holds for $\mathcal{A}$ and $\mathcal{X}$ and let p be any cluster ensemble map.

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- There is a choice of cluster structure for $\mathcal{A}^\vee$ and $\mathcal{X}^\vee$ such that $p^\vee : \mathcal{X}^\vee \to \mathcal{A}^\vee$ is again a cluster ensemble map.

Induced maps of mirrors

$p^\vee : \mathcal{X}^\vee \to \mathcal{A}^\vee$ as cluster ensemble map

In skew symmetric type, relevant choice of cluster structure is associated to the **chiral dual** initial data $(\Gamma^{\mathrm{op}}, \mathbf{s}^{\mathrm{op}})$:

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Then:

- $\mathcal{X}^\vee = \mathcal{A}_{\Gamma^{\mathrm{op}},[\mathbf{s}^{\mathrm{op}}]}$

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- $\mathcal{X}^\vee = \mathcal{A}_{\Gamma^{\mathrm{op}},[\mathbf{s}^{\mathrm{op}}]}$
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Induced maps of mirrors

$p^\vee : \mathcal{X}^\vee \to \mathcal{A}^\vee$ as cluster ensemble map

In skew symmetric type, relevant choice of cluster structure is associated to the **chiral dual** initial data $(\Gamma^{\mathrm{op}}, \mathbf{s}^{\mathrm{op}})$:

- $N_{\Gamma^{\mathrm{op}}} = N_\Gamma$, with skew-form $\{\cdot, \cdot\}_{\Gamma^{\mathrm{op}}} = -\{\cdot, \cdot\}_\Gamma$
- $(N_{\mathrm{uf}})_{\Gamma^{\mathrm{op}}} = (N_{\mathrm{uf}})_\Gamma$
- $I_{\Gamma^{\mathrm{op}}} = I_\Gamma$ and $(I_{\mathrm{uf}})_{\Gamma^{\mathrm{op}}} = (I_{\mathrm{uf}})_\Gamma$
- $\mathbf{s}^{\mathrm{op}} = \mathbf{s}$

Then:

- $\mathcal{X}^\vee = \mathcal{A}_{\Gamma^{\mathrm{op}}, [\mathbf{s}^{\mathrm{op}}]}$
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- $(p^\vee)^* : n \mapsto (p^*)^*(n)$

Grassmannian cluster structure

Plabic graphs for $\mathrm{Gr}_{n-k}(\mathbb{C}^n)$ ([Pos06], [RW19])

- A **plabic graph** G is an undirected graph drawn on a disk with cyclically ordered boundary vertices $(1, \cdots, n)$, and each internal vertex either black or white.

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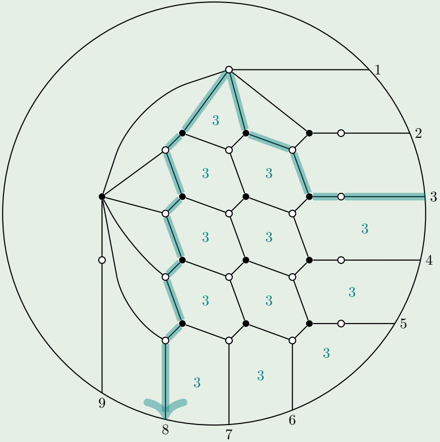
- The **trip** T_i is the path from i to some boundary vertex $\pi_G(i)$ consisting of maximal right turns at black vertices and maximal left turns at white vertices.
- If G is a "reduced" plabic graph and $\pi_G(i) = i + (n-k)$ for all i– "G is of **type** $\pi_{k,n}$"– then the trips assign Plücker labels in $\binom{[n]}{n-k}$ to each face as illustrated in the following example.

Grassmannian cluster structure

Example ($G^{\mathrm{rec}}_{4,9}$)



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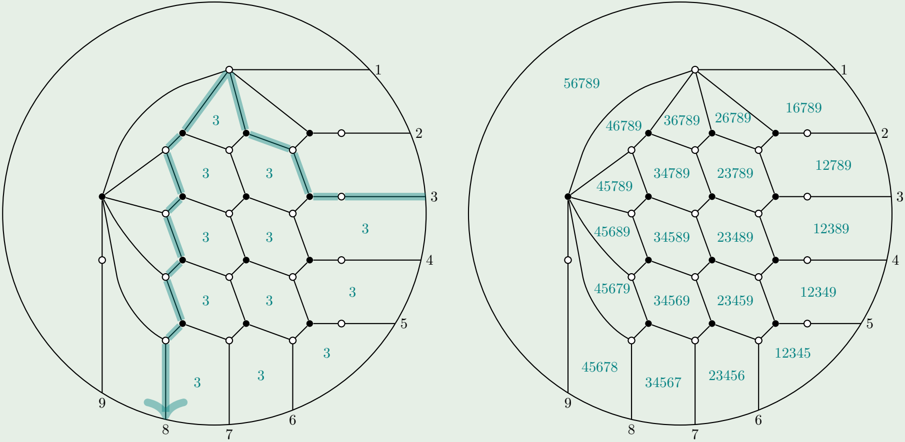

Grassmannian cluster structure

$\mathcal{A}$ cluster structure ([Sco06])

The corresponding Plücker coordinates form a cluster in Scott's $\mathcal{A}$-cluster structure of $\mathrm{UT}_{\mathrm{Gr}_{n-k}(\mathbb{C}^n)}{}^{\circ} := \mathrm{UT}_{\mathrm{Gr}_{n-k}(\mathbb{C}^n)} \setminus D$. Here,

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Grassmannian cluster structure

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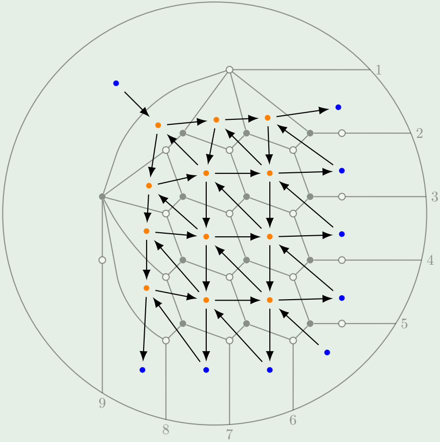

Grassmannian cluster structure

Plabic graphs for $\mathrm{Gr}_k(\mathbb{C}^n)$

Let G^{op} be the plabic graph obtained by swapping colors of all internal vertices of G.

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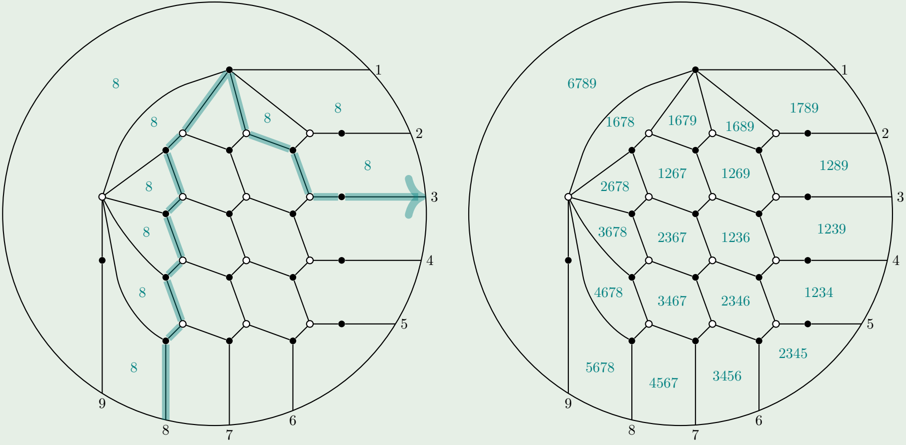
- The Plücker indices associated to the faces of G and G^{op} are related by $J \mapsto \pi_G(J)^c$.
- We have $Q(G)^{\mathrm{op}} = Q(G^{\mathrm{op}})$– so $(\Gamma, \mathbf{s}) \mapsto (\Gamma^{\mathrm{op}}, \mathbf{s}^{\mathrm{op}})$.

Grassmannian cluster structure

Example ($(G_{4,9}^{\mathrm{rec}})^{\mathrm{op}}$)



Grassmannian cluster structure

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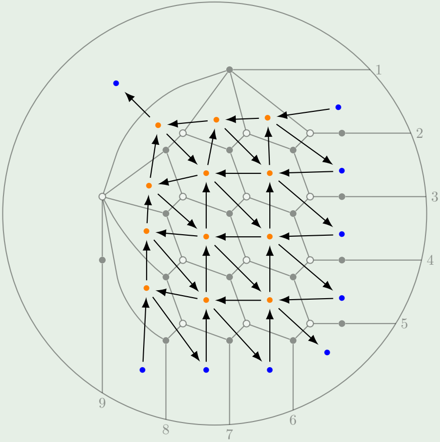

Grassmannian cluster structure

Two $\mathcal{X}$-cluster structures

- If $\mathcal{A}_{\Gamma,[\mathbf{s}]}$ is the $\mathcal{A}$-cluster variety in $\mathrm{UT}_{\mathrm{Gr}_{n-k}(\mathbb{C}^n)}$, then the same initial data determines a cluster variety $\mathcal{X}_{\Gamma,[\mathbf{s}]}$.

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- A plabic graph G of type $\pi_{k,n}$ also determines an $\mathcal{X}$ variety $\mathcal{X}^{\mathrm{net}}_{[G]}$ explicitly embedded in $\mathrm{Gr}_{n-k}(\mathbb{C}^n)$ in terms of *network parameters*.

Grassmannian cluster structure

Definition ([RW19])

- A **perfect orientation** $\mathcal{O}$ of a plabic graph G is an orientation of its edges such that every black internal vertex has exactly one outgoing edge and every white internal vertex has exactly one incoming edge.

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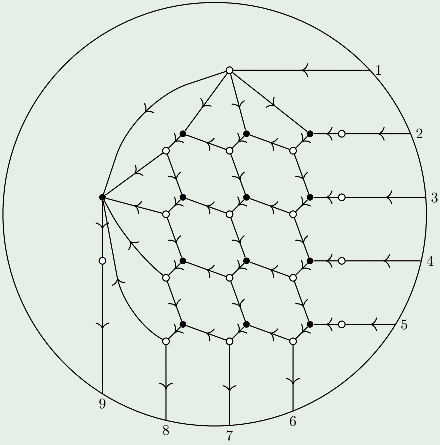
- To each face v of G, associate a **network parameter** x_v.
- Let the **weight of a path** ρ, denoted $\mathrm{wt}(\rho)$, be the product of all network parameters x_v for v a face to the left of ρ, and let the **weight of a flow** F– $\mathrm{wt}(F)$– be the product of the weights of all paths ρ in F.

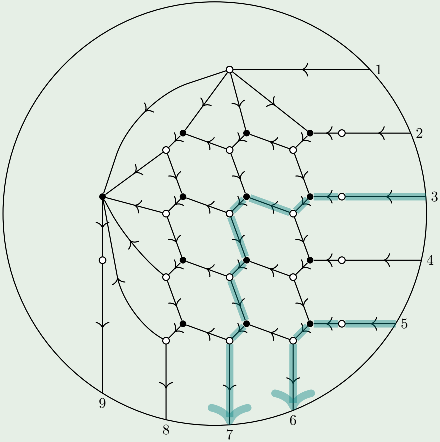
Grassmannian cluster structure

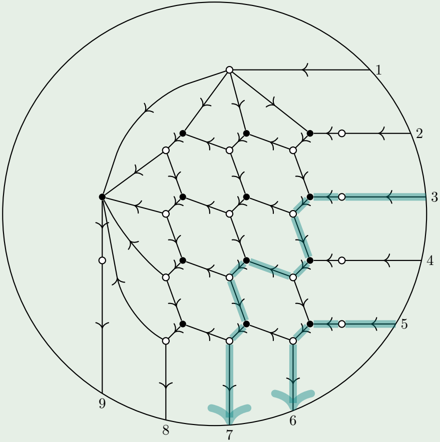
Example (Perfect orientation of $G_{4,9}^{\text{rec}}$ with $I_{\mathcal{O}} = [1,5]$)



Grassmannian cluster structure

Example (The flows from $I_{\mathcal{O}}$ to $\{1, 2, 4, 6, 7\}$)





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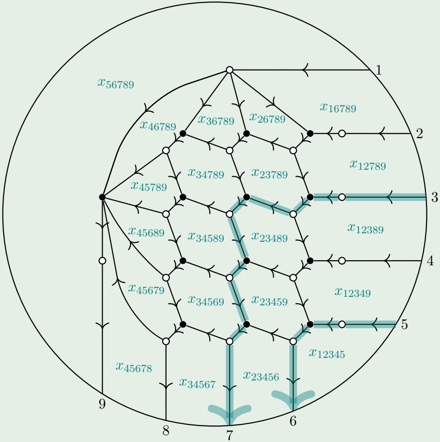
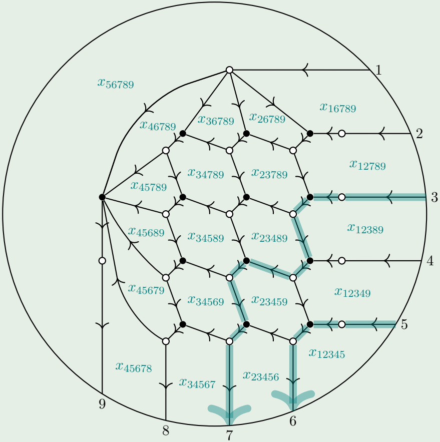


$x_{12345}^2 x_{12349} x_{12389} x_{23456} x_{23459} x_{23489}$



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$$T_{G,\mathcal{O}} := \mathrm{Spec}\left(\mathbb{C}[x_v^{\pm 1} : v \in \mathcal{P}_G, \prod_{v \in \mathcal{P}_G} x_v = 1]\right)$$

embeds into the affine open set where $p_{I_{\mathcal{O}}}$ is non-zero via **flow polynomials**.

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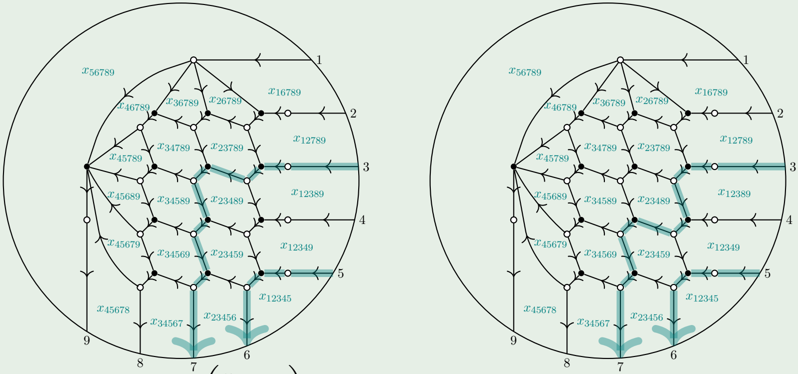
embeds into the affine open set where $p_{I_\mathcal{O}}$ is non-zero via **flow polynomials**.
Let $\mathcal{F}_{G,\mathcal{O}}(J)$ be the set of flows from $I_\mathcal{O}$ to J.

$$\text{Flow}_{G,\mathcal{O}}\left(\frac{p_J}{p_{I_\mathcal{O}}}\right) := \sum_{F \in \mathcal{F}_{G,\mathcal{O}}(J)} \text{wt}(F)$$

Grassmannian cluster structure

Example



$$\mathrm{Flow}_{G^{\mathrm{rec}}_{4,9},\mathcal{O}}\left(\frac{p_{12467}}{p_{12345}}\right) = x_{12345}^2 x_{12349} x_{12389} x_{23456} x_{23459} x_{23489}$$

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Remark

All constructions we have described for $\mathrm{Gr}_{n-k}(\mathbb{C}^n)$ apply to $\mathrm{Gr}_k(\mathbb{C}^n)$ as well. In fact, $[(G^{\mathrm{rec}}_{k,n})^{\mathrm{op}}] = [G^{\mathrm{rec}}_{n-k,n}]$.

Landau-Ginzburg mirror families for Grassmannians

Gross-Hacking-Keel(-Kontsevich) perspective

- Each $D_{[i+1,i+(n-k)]}$ defines a point $\mathrm{ord}_{D_{[i+1,i+(n-k)]}}$ in $(\mathrm{Gr}_{n-k}(\mathbb{C}^n)^\circ)^{\mathrm{trop}}(\mathbb{Z})$.

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- As such, it defines a ϑ-function $\vartheta_{\mathrm{ord}_{D_{[i+1,i+(n-k)]}}}$ on the mirror family $\mathcal{Y} \to T_{\mathrm{Cl}(\mathrm{Gr}_{n-k}(\mathbb{C}^n))}$.

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- If the frozen vertex v associated to $p_{[i+1,i+(n-k)]}$ is a source of $Q_{\Gamma,\mathbf{s}}$, then $\vartheta_{\mathrm{ord}_{D_{[i+1,i+(n-k)]}}}\Big|_{T_{N;\mathbf{s}^{\mathrm{op}}}} = z^{-e_v}$.

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- Plücker coordinates are $\mathcal{A}$ variables, so view $\mathrm{Gr}_{n-k}(\mathbb{C}^n)$ as a compactification of an $\mathcal{X}$ variety and the potential as a function on an $\mathcal{A}$ variety.

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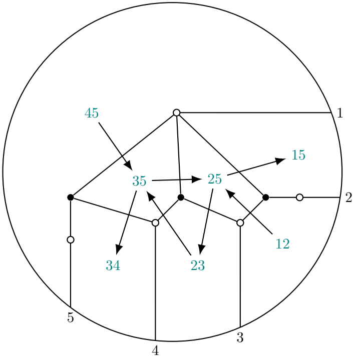
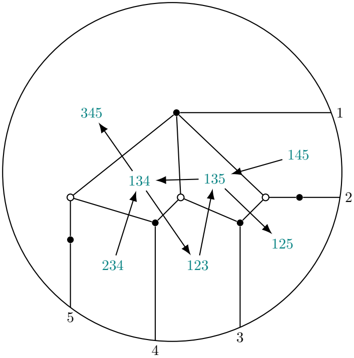
If $(k,n) \notin \{(2,4),(1,n),(n-1,n)\}$, this pair of maps is unique and both automorphisms are given in terms of pullbacks by $p_J \mapsto p_{J-|J|}$.

Example ($k = 3$, $n = 5$)





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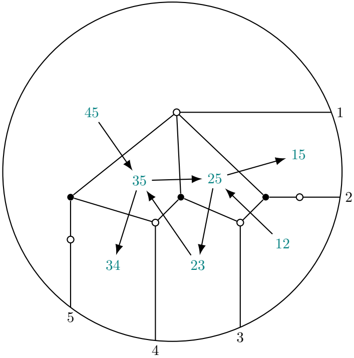

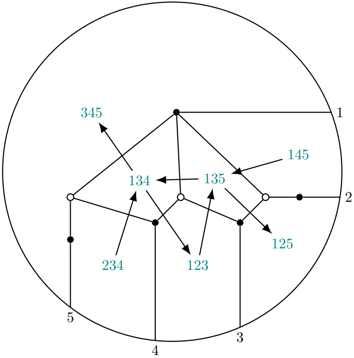

$$W_{\vartheta}^{2,5} = \sum_{i=1}^{5} \vartheta_{\mathrm{ord}_{D_{[i+1,i+3]}}}$$

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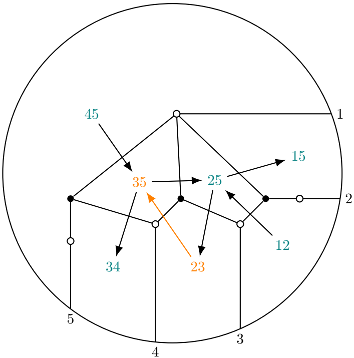
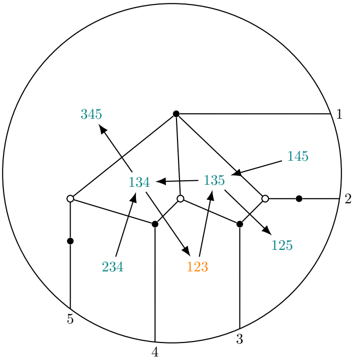
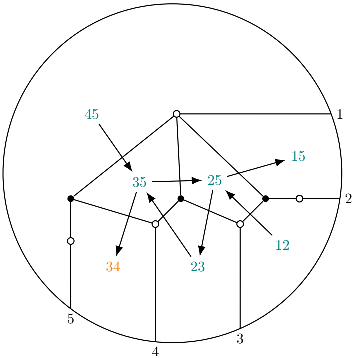


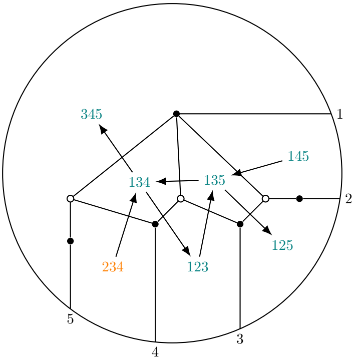


$$W_{\vartheta}^{2,5} = \sum_{i=1}^{5} \vartheta_{\mathrm{ord}_{D_{[i+1,i+3]}}}$$

$$\vartheta_{\mathrm{ord}_{D_{123}}}\Big|_{T_{N;\mathbf{s}}} = z^{-e_{23}} + z^{-e_{23}-e_{35}}$$

Example ($k = 3$, $n = 5$)

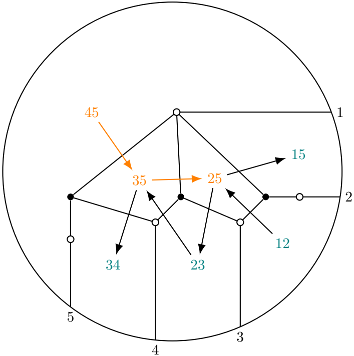


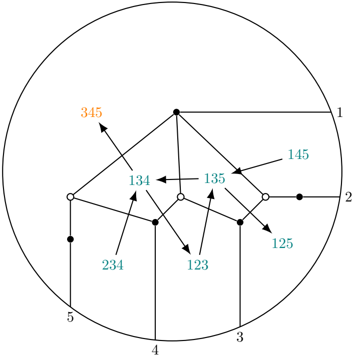


$$W_\vartheta^{2,5} = \sum_{i=1}^{5} \vartheta_{\mathrm{ord}_{D_{[i+1,i+3]}}}$$

$$\vartheta_{\mathrm{ord}_{D_{234}}}\Big|_{T_{N;\mathbf{s}}} = z^{-e_{34}}$$

Example ($k = 3$, $n = 5$)





$$W_{\vartheta}^{2,5} = \sum_{i=1}^{5} \vartheta_{\mathrm{ord}_{D_{[i+1,i+3]}}}$$

$$\vartheta_{\mathrm{ord}_{D_{345}}}\Big|_{T_{N;\mathbf{s}}} = z^{-e_{45}} + z^{-e_{45}-e_{35}} + z^{-e_{45}-e_{35}-e_{25}}$$

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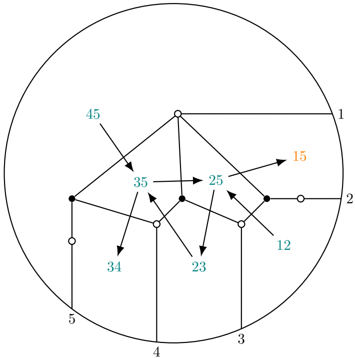

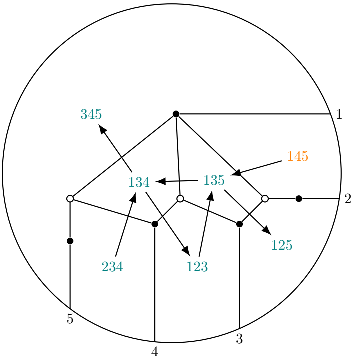

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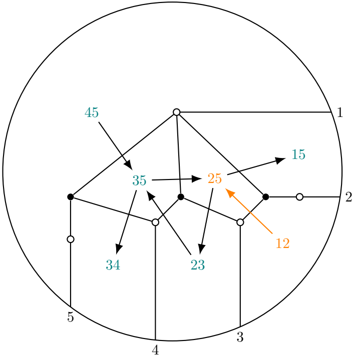
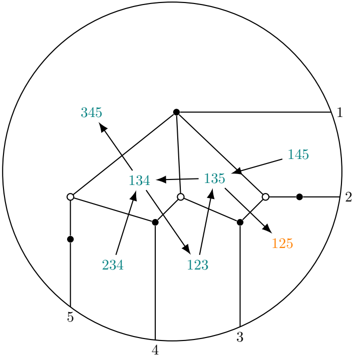
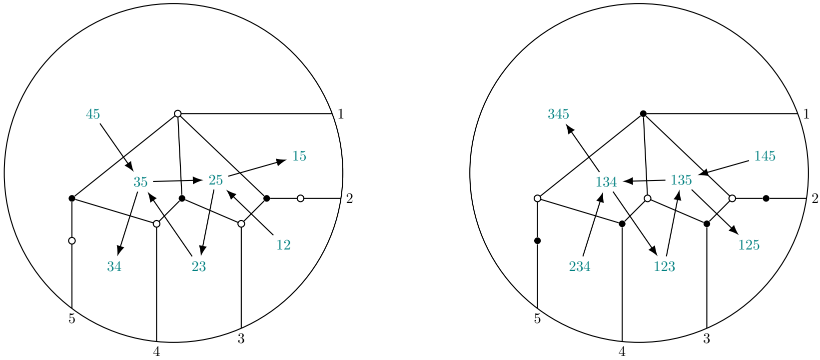




$$W_{\vartheta}^{2,5} = \sum_{i=1}^{5} \vartheta_{\mathrm{ord}_{D_{[i+1,i+3]}}}$$

$$\vartheta_{\mathrm{ord}_{D_{125}}}\Big|_{T_{N;\mathbf{s}}} = z^{-e_{12}} + z^{-e_{12}-e_{25}}$$

Example ($k = 3$, $n = 5$)



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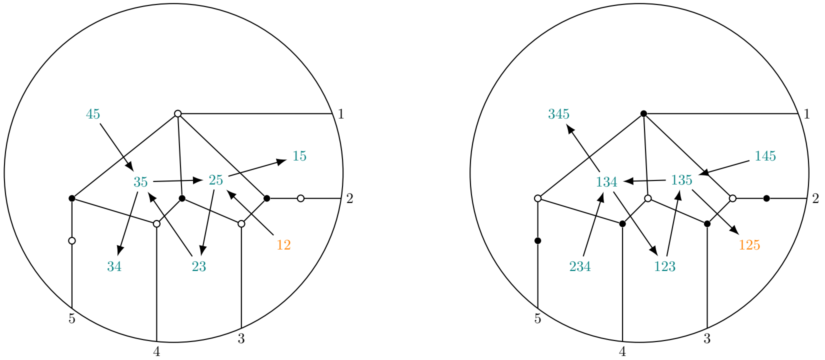

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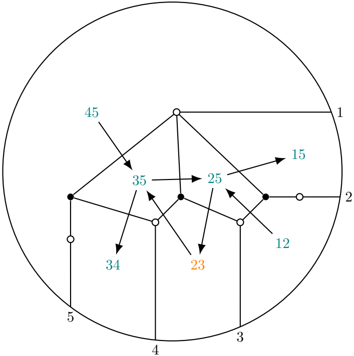

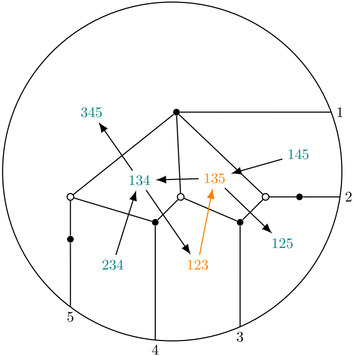

$$W_{\vartheta}^{3,5} = \sum_{i=1}^{5} \vartheta_{\mathrm{ord}_{D_{[i+1,i+2]}}}$$

$$\vartheta_{\mathrm{ord}_{D_{23}}}\Big|_{T_{N;\mathbf{s}^{\mathrm{op}}}} = z^{-e_{123}} + z^{-e_{123}-e_{135}}$$

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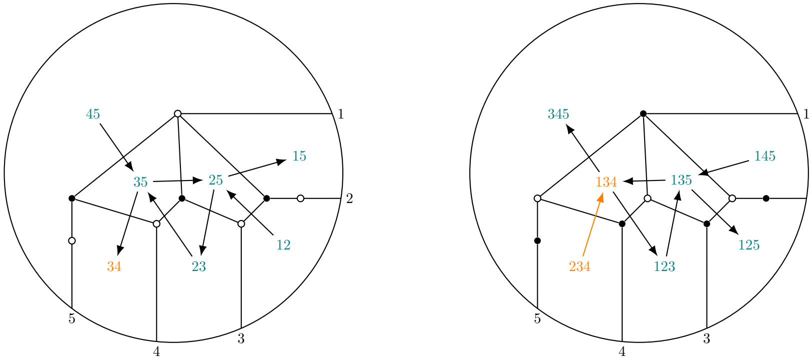

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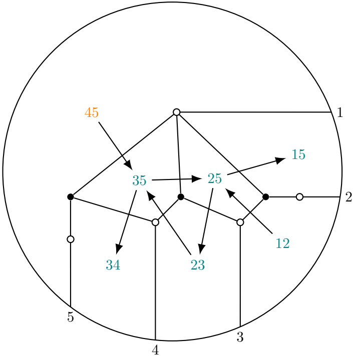

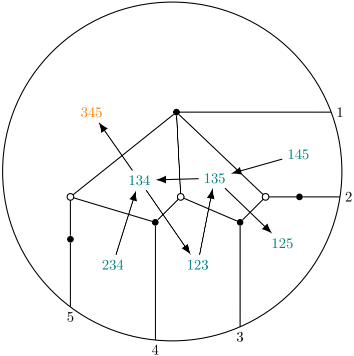

$$W_{\vartheta}^{3,5} = \sum_{i=1}^{5} \vartheta_{\mathrm{ord}_{D_{[i+1,i+2]}}}$$

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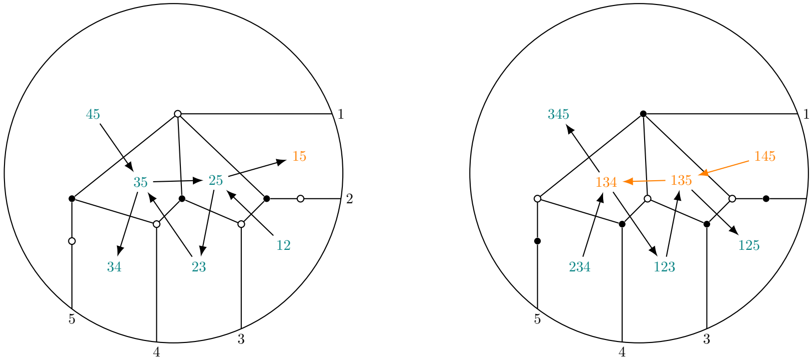

$$W_{\vartheta}^{3,5} = \sum_{i=1}^{5} \vartheta_{\mathrm{ord}_{D_{[i+1,i+2]}}}$$

$$\vartheta_{\mathrm{ord}_{D_{15}}}\Big|_{T_{N;\mathbf{s}^{\mathrm{op}}}} = z^{-e_{145}} + z^{-e_{145}-e_{135}} + z^{-e_{145}-e_{135}-e_{134}}$$

Example (Potentials on $\mathrm{Gr}_2\left(\mathbb{C}^5\right)^\circ$)

- $\left.\vartheta_{\mathrm{ord}_{D_{123}}}\right|_{T_{N;\mathbf{s}}} = z^{-e_{23}} + z^{-e_{23}-e_{35}}$

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- $\vartheta_{\mathrm{ord}_{D_{123}}}\Big|_{T_{N;\mathbf{s}}} = z^{-e_{23}} + z^{-e_{23}-e_{35}}$
- $p^\vee(D_{123}) = D_{145} \rightsquigarrow W_q^{2,5}$ summand $\frac{p_{24}}{p_{23}}$

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- $p^*(-e_{23}) \in e_{25}^* - e_{35}^* + N_{\mathrm{uf}}^\perp$ and $p^*(-e_{35}) = e_{23}^* + e_{45}^* - e_{25}^* - e_{34}^*$

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- $p^*(-e_{23}) \in e^*_{25} - e^*_{35} + N_{\mathrm{uf}}^\perp$ and $p^*(-e_{35}) = e^*_{23} + e^*_{45} - e^*_{25} - e^*_{34}$
- With $p^*(-e_{23}) = e^*_{25} + e^*_{34} - e^*_{23} - e^*_{35}$, we get

$$p^*\left(z^{-e_{23}} + z^{-e_{23}-e_{35}}\right) = z^{e^*_{25}+e^*_{34}-e^*_{23}-e^*_{35}} + z^{e^*_{45}-e^*_{35}},$$

so $p^*\left(\vartheta_{\mathrm{ord}_{D_{123}}}\right) = \frac{p_{24}}{p_{23}}$.

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- With $p^*(-e_{23}) = e_{25}^* + e_{34}^* - e_{23}^* - e_{35}^*$, we get

$$p^*(z^{-e_{23}} + z^{-e_{23}-e_{35}}) = z^{e_{25}^*+e_{34}^*-e_{23}^*-e_{35}^*} + z^{e_{45}^*-e_{35}^*},$$

  so $p^*(\vartheta_{\mathrm{ord}_{D_{123}}}) = \frac{p_{24}}{p_{23}}$.
- Other summands similar.

Example (Potentials on $\mathrm{Gr}_3\left(\mathbb{C}^5\right)^\circ$)

- $\vartheta_{\mathrm{ord}_{D_{12}}}\Big|_{T_{N;\mathbf{s}^{\mathrm{op}}}} = z^{-e_{125}}$

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- $p(D_{12}) = D_{34} \rightsquigarrow \frac{p_{135}}{p_{125}}$
- $(p^\vee)^*(-e_{125}) \in e^*_{135} + N^\perp_{\mathrm{uf}}$
- With $(p^\vee)^*(-e_{125}) = e^*_{135} - e^*_{125}$, we get $(p^\vee)^*(\vartheta_{\mathrm{ord}_{D_{12}}}) = \frac{p_{135}}{p_{125}}$.
- Other summands similar.

Connection to [RW19]

Corollary (Bossinger, Cheung, M, Nájera Chávez)

Identification of superpotential polytopes for $\mathrm{Gr}_{n-k}(\mathbb{C}^n)$:
Fix positive constants c_i for $i \in [1, n]$. Let

$$P = \bigcap_i \left\{ x \in (\mathcal{A}^\vee_{\mathbf{1} \in T_K})^{\mathrm{trop}}(\mathbb{R}) : \vartheta^{\mathrm{trop}}_{D_{[i+1, i+(n-k)]}}(x) \geq -c_i \right\}$$

and

$$Q = \bigcap_i \left\{ a \in (\mathcal{X}^\vee / T_{K^\vee})^{\mathrm{trop}}(\mathbb{R}) : \left(\frac{p_{[i+1, i+(n-k)] \cup \{i+k+1\}}}{p_{[i+1, i+k]}} \right)^{\mathrm{trop}} (a) \geq -c_{i+2k} \right\}.$$

Then $(\overline{p}^\vee)^{\mathrm{trop}}(Q) = P$.

Connection to [RW19]

Rietsch-Williams use $\mathcal{X}^{\mathrm{net}}$ coordinates to describe their NO bodies and toric degenerations. So:

Connection to [RW19]

Theorem (Bossinger, Cheung, M, Nájera Chávez)

The Plücker coordinates whose flow polynomials with respect to $((G^{\mathrm{rec}}_{k,n})^{\mathrm{op}}, \mathcal{O})$ are monomials form precisely the $\mathcal{A}$ cluster of $G^{\mathrm{rec}}_{n-k,n}$.

Connection to [RW19]

Theorem (Bossinger, Cheung, M, Nájera Chávez)

The Plücker coordinates whose flow polynomials with respect to $((G^{\mathrm{rec}}_{k,n})^{\mathrm{op}}, \mathcal{O})$ are monomials form precisely the $\mathcal{A}$ cluster of $G^{\mathrm{rec}}_{n-k,n}$. There is an isomorphism $\psi : \mathcal{X}^{\mathrm{net}}_{[(G^{\mathrm{rec}}_{k,n})^{\mathrm{op}}]} \to \mathcal{X}_{[G^{\mathrm{rec}}_{n-k,n}],\mathbf{1}}$ that is a monomial transformation which identifies $\mathcal{X}$ variables for mutable indices and gives the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{X}^{\mathrm{net}}_{[(G^{\mathrm{rec}}_{k,n})^{\mathrm{op}}]} & \xrightarrow{\ \mathrm{Flow}_{(G^{\mathrm{rec}}_{k,n})^{\mathrm{op}},\mathcal{O}}\ } & \mathcal{A}_{[G^{\mathrm{rec}}_{n-k,n}]}/T_K \\
 & \searrow^{\psi} & \downarrow{\scriptstyle p_{[G^{\mathrm{rec}}_{n-k,n}]}} \\
 & & \mathcal{X}_{[G^{\mathrm{rec}}_{n-k,n}],\mathbf{1}}
\end{array}$$

Connection to [RW19]

Using ψ we can recover the Rietsch-Williams NO bodies and toric degenerations as well.

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