Convexity in tropical spaces and compactifications of cluster varieties

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Joint work with Man-Wai Cheung and Alfredo Nájera Chávez

arXiv:1912.13052 [math.AG]

Goal and Background

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Generalize the polytope construction of projective **toric varieties** to the non-toric world

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- One of the simplest classes of objects in algebraic geometry– very amenable to computations and proofs

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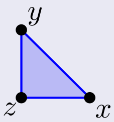
Toric Picture:

A d-dimensional convex rational polytope defines a d-dimensional (polarized) projective toric variety.

- Integral points of polytope $P \rightsquigarrow$ Sections of line bundle $\mathcal{L}$
- Integral points of dilations of $P \rightsquigarrow$ Sections of powers of $\mathcal{L}$

Convexity is key!

Example



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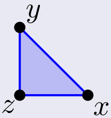

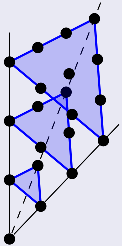

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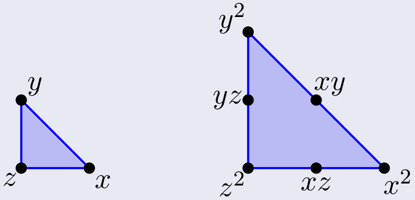

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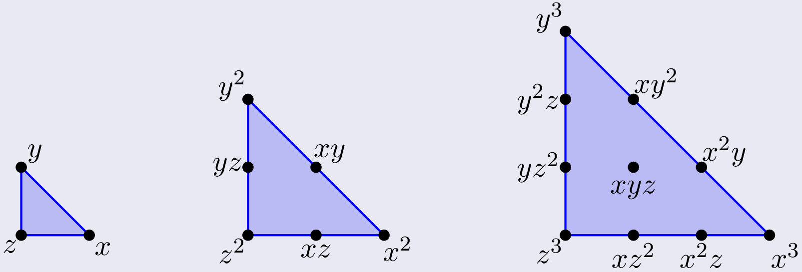

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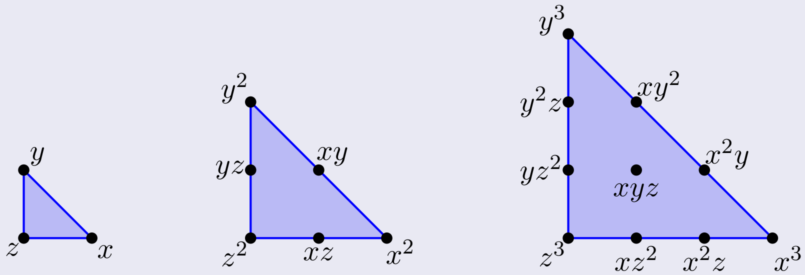
Example $(\mathbb{P}^2, \mathcal{O}(1))$



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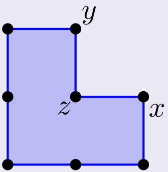

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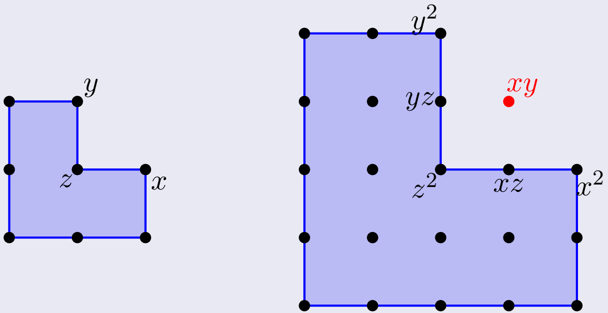

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Let $T \cong (\mathbb{C}^*)^n$ and $M = \mathrm{char}(T)$.

Want cluster analogue of this:

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Definition (*Log Calabi-Yau variety*)

A smooth complex variety U with a unique volume form Ω having at worst a simple pole along any divisor in *any* compactification of U

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Fact: If (Y, D) is any toric variety with its toric boundary divisor, then Ω has a simple pole along each component of D.

Cluster Varieties: Context and Definition

Example (Carefully glued tori)

$$U = \bigcup_i T_i / \sim$$

$$\mu_{ij} : T_i \dashrightarrow T_j, \qquad \mu_{ij}^* (\Omega_j) = \Omega_i$$

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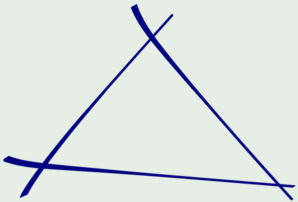

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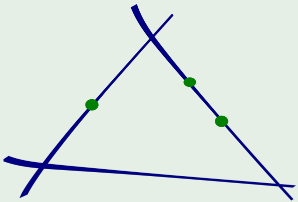

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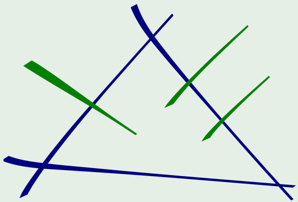

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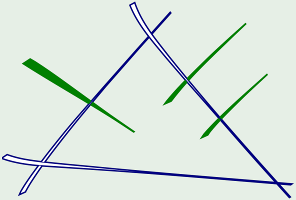

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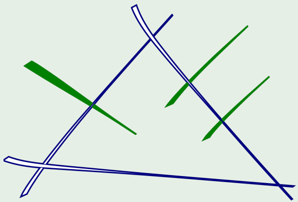
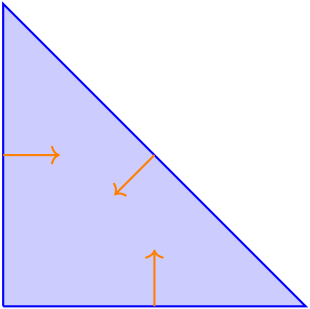


Two ways of describing a **Cluster Variety**

The $\mathbb{P}^2$ example



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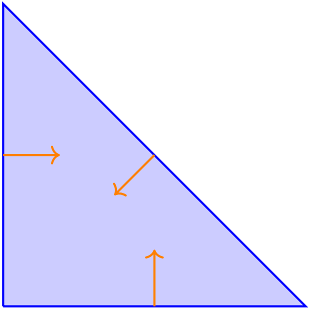

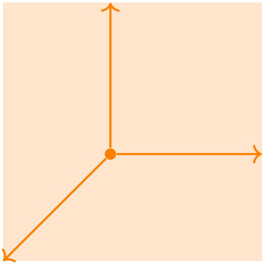

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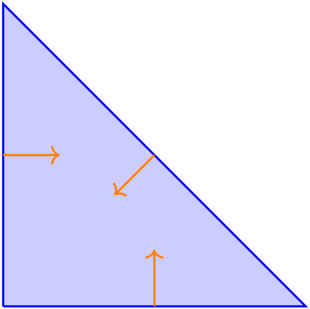

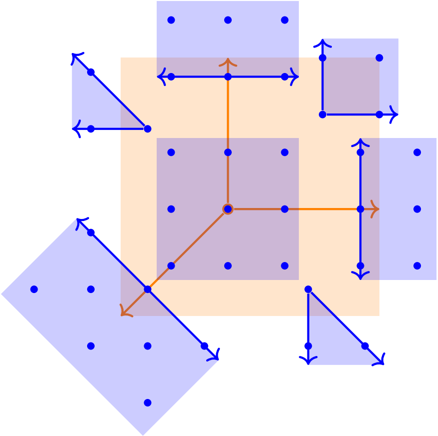

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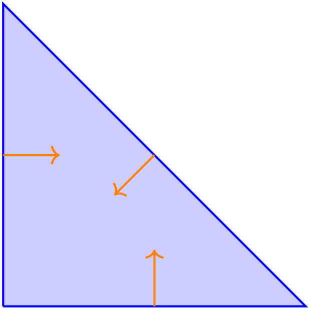

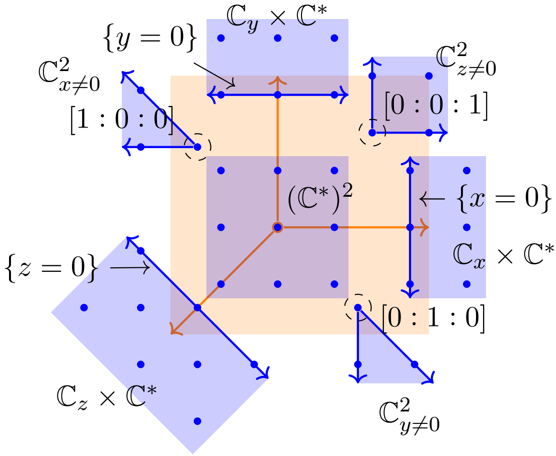

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- A d-dimensional cone corresponds to a codimension d stratum

Tropicalization of a log Calabi-Yau variety

Definition

Let (U, Ω) be log CY. A **divisorial discrete valuation** (ddv) $\nu : \mathbb{C}(U) \setminus \{0\} \to \mathbb{Z}$ is a discrete valuation of the form $\nu = \mathrm{ord}_D(\,\cdot\,)$ where D is (a positive multiple of) an irreducible effective divisor in a variety birational to U.

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If $U = T_N$, $U^{\mathrm{trop}}(\mathbb{Z}) = N$. Recall that toric divisors are indexed by cocharacters.

Tropicalization of a log Calabi-Yau variety

Remark

- We can extend scalars from $\mathbb{Z}_{>0}$ to $\mathbb{R}_{>0}$ in the definition of $U^{\mathrm{trop}}(\mathbb{Z})$ to obtain $U^{\mathrm{trop}}(\mathbb{R})$ – the **real tropicalization of** U.

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- $U^{\text{trop}}(\mathbb{R})$ has a natural piecewise linear structure.
- When $U = T_N$, $U^{\text{trop}}(\mathbb{R}) = N_{\mathbb{R}}$ is actually linear.

Scattering Diagrams (from Gross-Siebert program)

Rough definition

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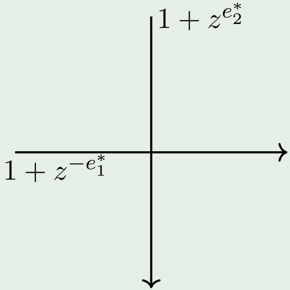
- In cluster case [GHKK18] give algorithm for building scattering diagram from simple initial data.

Scattering Diagrams (from Gross-Siebert program)

Example

Initial Data:

- $N = \mathbb{Z}^2$
- $\mathbf{s} = (e_1, e_2)$
- $\{e_1, e_2\} = 1$



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- Scattering function f on wall $\mathfrak{d} \subset n^{\perp}$ defines wall-crossing map $\mathfrak{p}_f : z^m \mapsto z^m f^{\pm\langle m,n\rangle}$, with sign determined by crossing direction.

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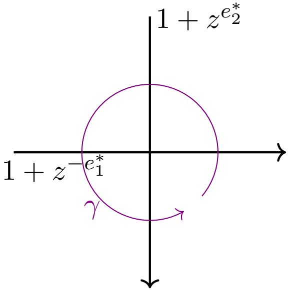

$$z^{e_1^*} \mapsto z^{e_1^*}\left(1 + z^{e_2^* - e_1^*}\right)$$

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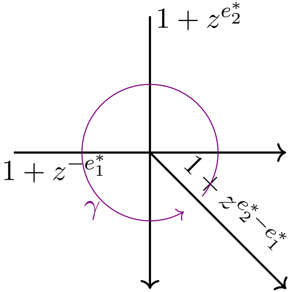


$z^m \mapsto z^m$

Consistent

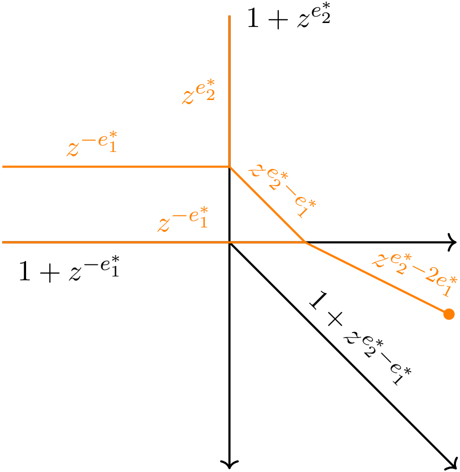
Scattering Diagrams (from Gross-Siebert program)

Theorem (Gross-Hacking-Keel-Kontsevich, using 2D result of Kontsevich-Soibelman)

There is a unique (up to equivalence) consistent scattering diagram associated to every cluster variety.

Broken Lines

Example



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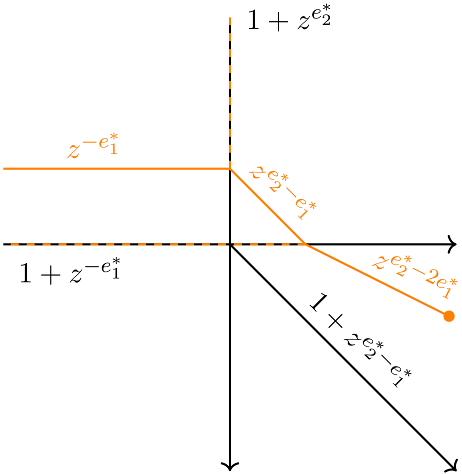

ϑ-functions

From broken lines to ϑ-functions

- Get "ϑ-function" on $U^\vee$ for each $p \in U^{\mathrm{trop}}(\mathbb{Z})$– think N is a basis for $\mathcal{O}(T_M = T_N^\vee)$.

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  - Write ϑ_p as sum of final decorating monomials for broken lines with initial monomial z^p and endpoint Q.

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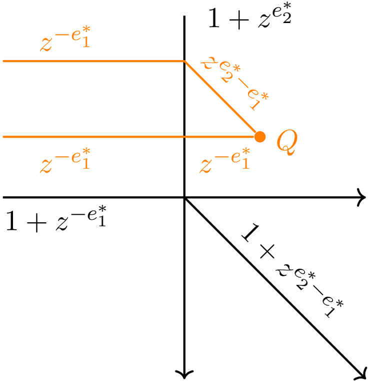
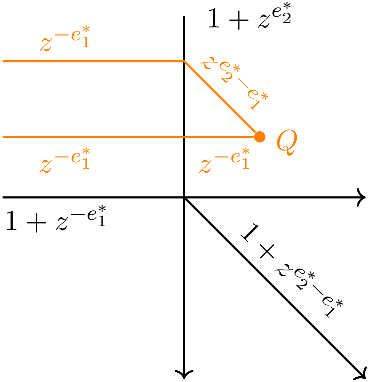


ϑ-functions

$$\vartheta_{-e_1^*} = z^{e_2^*-e_1^*} + z^{-e_1^*}$$



ϑ-functions

Remark

Consistency ensures that the local expressions for ϑ_p patch together to give a global function.

ϑ-function multiplication

Structure constants α_{pq}^r

$$\vartheta_p \cdot \vartheta_q = \sum_{r \in U^{\mathrm{trop}}(\mathbb{Z})} \alpha_{pq}^r \vartheta_r$$

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Theorem (Gross-Hacking-Keel-Kontsevich)

$$\alpha_{pq}^r = \sum_{\substack{(\gamma_1,\gamma_2) \\ I(\gamma_1)=p,\ I(\gamma_2)=q \\ \gamma_1(0)=\gamma_2(0)=r \\ F(\gamma_1)+F(\gamma_2)=r}} c(\gamma_1)\ c(\gamma_2)$$

ϑ-function multiplication

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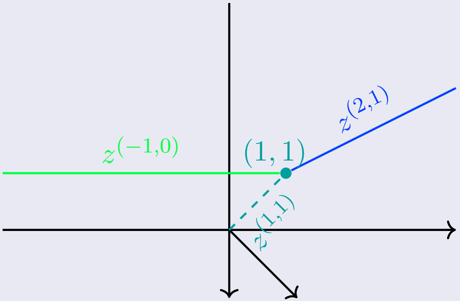

$$\vartheta_{(-1,0)} \cdot \vartheta_{(2,1)} = \vartheta_{(1,1)} +$$

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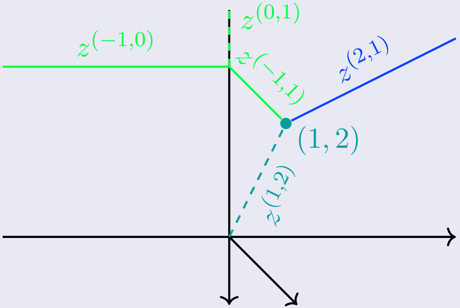

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So far have:

- $U^{\text{trop}}(\mathbb{Z})$ "is" ϑ-basis for $\mathcal{O}(U^{\vee})$

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Let's make this more precise.

Definition (Positive subset)

A closed subset $S \subset U^{\mathrm{trop}}(\mathbb{R})$ is **positive** if for every $a, b \in \mathbb{Z}_{\geq 0}$, $p \in aS(\mathbb{Z})$, $q \in bS(\mathbb{Z})$, and $r \in U^{\mathrm{trop}}(\mathbb{Z})$ with $\alpha^r_{p,q} \neq 0$ we have: $r \in (a+b)\, S$.

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When is S positive?

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Definition (Broken line convex [Cheung, M., Nájera Chávez])

A closed subset $S \subset U^{\text{trop}}(\mathbb{R})$ is **broken line convex** if for every $x, y \in S(\mathbb{Q})$, every broken line segment connecting x and y is entirely contained in S.

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Theorem (Cheung, M., Nájera Chávez)

S is positive if and only if S is broken line convex.

Key to proof

The problem

- Broken line convexity deals with **endpoints** of broken line segments.

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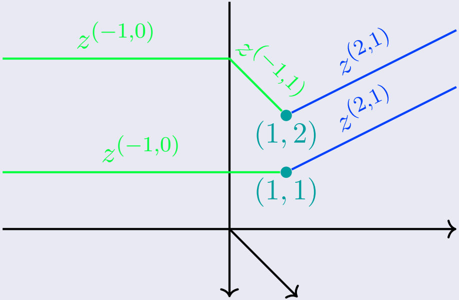
The fix

Give a "jagged path" ([GS16]) type description of ϑ-function multiplication– contributions come as weighted averages along broken line segments.

Key to proof

Example



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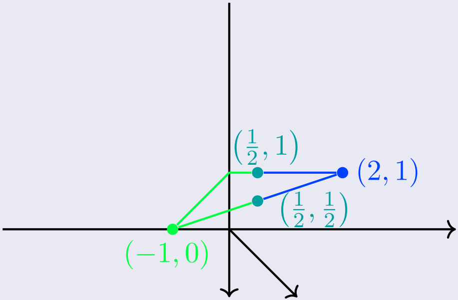

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The generalization:

- $U^{\mathrm{trop}}(\mathbb{Z})$ "is" ϑ-basis for $\mathcal{O}(U^{\vee})$
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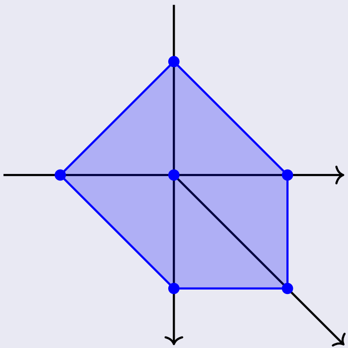
Examples

Example (Anticanonical "polytope" of degree 5 del Pezzo surface)



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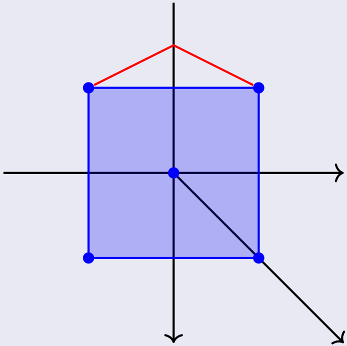
Non-example



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Newton-Okounkov bodies

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- Important tool for studying toric degenerations
- NO body genuinely depends on ν, not just the geometric input (X, D)

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- Each choice of torus chart identifies $\Delta_\vartheta(D)$ with a usual NO body

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- Analogous result holds for complete flag variety

Basic Definitions

Definition (Cartier divisor)

A divisor D on a normal variety X is **Cartier** if D is locally principal, meaning there is an open cover $\{U_i\}$ of X with $D|_{U_i}$ the divisor associated to zeros and poles of a rational function for all U_i.

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Definition (Gorenstein Fano variety)

A normal variety X is **Gorenstein Fano** if $-K_X$ is Cartier ($\rightsquigarrow$ Gorenstein) and ample ($\rightsquigarrow$ Fano).

Review of Toric Case

Toric anticanonical divisor

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Definition (Reflexive polytope)

A lattice polytope $P \subset M_{\mathbb{R}}$ is **reflexive** if its dual

$$P^\circ := \{n \in N_{\mathbb{R}} : \langle m, n \rangle \geq -1 \text{ for all } m \in P\}$$

is also a lattice polytope.

Review of Toric Case

Polytopes and toric Fanos

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Review of Toric Case

Batyrev's conjecture

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Batyrev Duality for Cluster Varieties?

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Level sets of W almost CY, but not compact. P° defines Gorenstein Fano $Y \supset T^\vee$, with W a section of the anticanonical bundle of Y.

The Cluster Case

Landau-Ginzburg model and anticanonical “polytope”

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- Restriction of evaluation pairing gives

$$\begin{aligned} \langle \cdot\,,\,\cdot \rangle : U^{\mathrm{trop}}(\mathbb{Z}) \times (U^\vee)^{\mathrm{trop}}(\mathbb{Z}) &\to \mathbb{Z} \\ (\,\nu\,,\,p\,) &\mapsto \nu(\vartheta_p) \end{aligned}$$

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- Define $\mathrm{Newt}_\vartheta(W) := \mathrm{conv}_{\mathrm{BL}}(\nu_i) \subset U^{\mathrm{trop}}(\mathbb{R})$.
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Guess: Generic sections of $\mathcal{O}_X(D)$ and $\mathcal{O}_Y(D')$ are mirror (mildly singular) Calabi-Yau varieties.

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  - When can we write ϑ_r as $\displaystyle\sum_{\substack{p \in aP(\mathbb{Z}) \\ q \in bP(\mathbb{Z})}} c_{p,q}\ \vartheta_p \cdot \vartheta_q$?

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EPSRC grant to pursue with Konstanze Rietsch

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$\Delta_\vartheta(D)^\circ$ defines Batyrev dual pair (Y, D') with $Y \setminus D' = U^\vee$.
Question: *How is the geometry of* (Y, D') *related to the representation theory of* (X, D)*?*

References

[Bat94] V. V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Alg. Geom , 493–535 (1994).

[GHKK18] M. Gross, P. Hacking, S. Keel and M. Kontsevich, *Canonical bases for cluster algebras*, J. Amer. Math. Soc. **31**(2), 497–608 (2018).

[GS16] M. Gross and B. Siebert, Theta functions and mirror symmetry, in *Surveys in differential geometry 2016. Advances in geometry and mathematical physics*, volume 21 of *Surv. Differ. Geom.*, pages 95–138, Int. Press, Somerville, MA, 2016.

[RW19] K. Rietsch and L. Williams, *Newton-Okounkov bodies, cluster duality, and mirror symmetry for Grassmannians*, Duke Math. J. **168**(18), 3437–3527 (2019).