Cluster varieties

Timothy Magee

After [GHK15a] and [GHKK18]

Log Calabi-Yau varieties

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Moral definition

A **log Calabi-Yau variety** is smooth complex variety U with a unique (up to scaling) volume form Ω having at worst a simple pole along any divisor in *any* compactification of U.

Log Calabi-Yau varieties

Fact (Follows from results of Iitaka)

Let (Y_1, D_1) and (Y_2, D_2) be a smooth projective variety Y_i with a normal crossing divisor D_i, such that $Y_1 \setminus D_1 = Y_2 \setminus D_2 =: U$. Then the subspaces $\Gamma(Y_1, \omega_{Y_1}(D_1)^{\otimes i}) \subset \Gamma(U, \omega_U^{\otimes i})$ and $\Gamma(Y_2, \omega_{Y_2}(D_2)^{\otimes i}) \subset \Gamma(U, \omega_U^{\otimes i})$ are the same for all i.

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Actual definition

A **log Calabi-Yau variety** is a smooth complex variety U such that for (Y, D) as above, the subspace $\Gamma(Y, \omega_Y(D)^{\otimes i}) \subset \Gamma(U, \omega_U^{\otimes i})$ is one dimensional and generated by $\Omega^{\otimes i}$ for all i for some volume form $\Omega \in \Gamma(U, \omega_U)$.

Examples

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Algebraic torus $U = T = (\mathbb{C}^*)^n$, $\Omega = \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$.

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Proof sketch:
Let $\phi \in \mathrm{SL}_n(\mathbb{Z})$. Then $\frac{dz^{\phi(e_1)}}{z^{\phi(e_1)}} \wedge \cdots \wedge \frac{dz^{\phi(e_n)}}{z^{\phi(e_n)}} = \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n} = \Omega$.
Divisorial components of the toric boundary are of the form $\{z^m = 0\}$ where m is a *primitive* element of the character lattice.
We can choose ϕ such that $\phi(e_1) = m$.

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Remark

If (U, Ω) is any log Calabi-Yau, the "interesting" compactifications mirror this toric example. A compactification (Y, D) of U is called a **minimal model** for U if Ω has a pole along each divisorial component of D.

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Remark

If (U, Ω) is any log Calabi-Yau, the "interesting" compactifications mirror this toric example. A partial compactification (Y, D) of U is called a **partial minimal model** for U if Ω has a pole along each divisorial component of D.

Examples

Example 2

Let $(\overline{Y}, \overline{D})$ be a toric variety, and let $H \subset D$ be a codimension 2 locus in the boundary. Now take (Y, D) to be the blow-up of $\overline{Y}$ along H together with the strict transform of $\overline{D}$. Then $U := Y \setminus D$ is log Calabi-Yau (with volume form the pullback of the toric volume form), and (Y, D) is a partial minimal model for U.

Examples

Example 3

Let U be a union of tori of the form

$$U = \bigcup_i T_i / \sim$$

$$\mu_{ij} : T_i \dashrightarrow T_j, \qquad \mu_{ij}^* (\Omega_j) = \Omega_i$$

Then the volume forms on each T_i patch together to give a global volume form and U is log Calabi-Yau.

Cluster varieties

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Relating the constructions in an example

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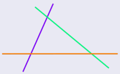

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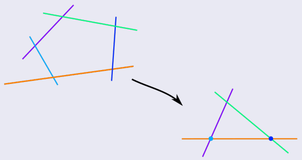

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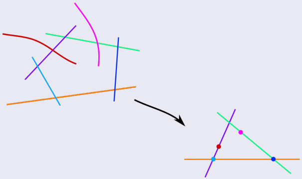

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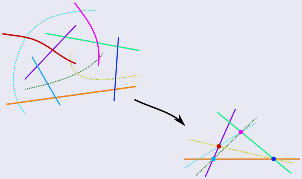

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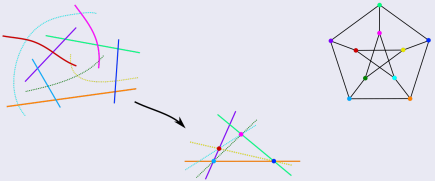

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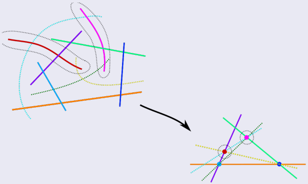

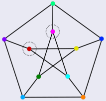

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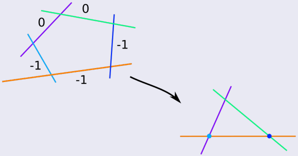

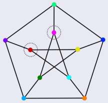

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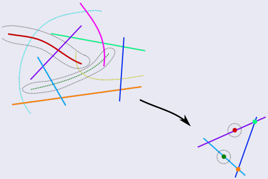

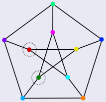

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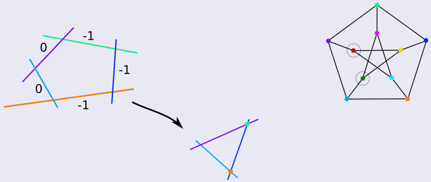

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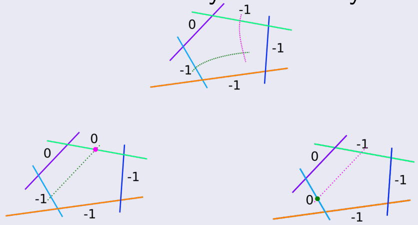

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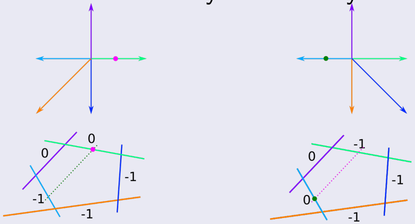

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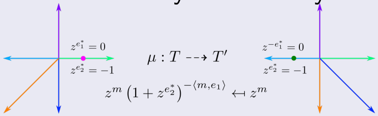

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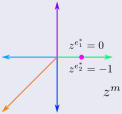

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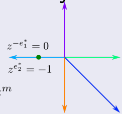

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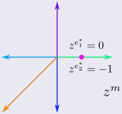
"Mutation"



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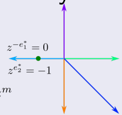

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The $\mathcal{A}$-variety as a cluster log CY

Set $\Sigma_{\mathbf{s},\mathcal{A}} := \{\mathbb{R}_{\geq 0} \cdot e_i : 1 \leq i \leq n\} \cup \{0\}$, and let $v_i := \{e_i, \cdot\} \in M$. Blow-up each D_{e_i} along $\{1 + z^{v_i} = 0\}$.

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The $\mathcal{X}$-variety as cluster log CY

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More precise definitions

The $\mathcal{A}$-variety as a union

Mutation $\mu_k : T_{N;\mathbf{s}} \dashrightarrow T_{N;\mathbf{s}'}$ defined in terms of pullback of functions: $\mu_k^*(z^m) = z^m \left(1 + z^{v_k}\right)^{-\langle m, e_k \rangle}$. Now set

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Let U be an affine log Calabi-Yau with *maximal boundary* – this means it has a minimal model (Y, D) where D has a 0-stratum.

Mirror conjecture

Conjecture of Gross-Hacking-Keel [GHK15b]

Let U be an affine log Calabi-Yau with *maximal boundary*. Then the mirror $U^\vee$ is again an affine log Calabi-Yau with maximal boundary. The integral tropical points of U parametrize a basis of ϑ-functions on $U^\vee$, with multiplication given explicitly in terms of broken line counts.

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There are multiple precise log Calabi-Yau mirror symmetry conjectures in arXiv version 1 of [GHK15b].

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Tropicalization

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Let (U, Ω) be log CY. A **divisorial discrete valuation** (ddv) $\nu : \mathbb{C}(U) \setminus 0 \to \mathbb{Z}$ is a discrete valuation of the form $\nu = \mathrm{ord}_D(\,\cdot\,)$ where D is (a positive multiple of) an irreducible effective divisor in a variety birational to U.

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If $U = T_N$, $U^{\text{trop}}(\mathbb{Z}) = N$.

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If $U = T_N$, $U^{\mathrm{trop}}(\mathbb{Z}) = N$. Recall that toric divisors are indexed by cocharacters.

Tropicalization

Remark

- We can extend scalars from $\mathbb{Z}_{>0}$ to $\mathbb{R}_{>0}$ in the definition of $U^{\mathrm{trop}}(\mathbb{Z})$ to obtain $U^{\mathrm{trop}}(\mathbb{R})$ – the **real tropicalization of** U.

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- $U^{\text{trop}}(\mathbb{R})$ has a natural piecewise linear structure.
- When $U = T_N$, $U^{\text{trop}}(\mathbb{R}) = N_{\mathbb{R}}$ is actually linear.

Scattering Diagrams and Broken Lines

Rough definition

A **scattering diagram** is a collection of *walls* in $U^{\mathrm{trop}}(\mathbb{R})$.

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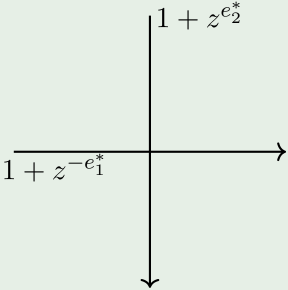
Example 5

- $N = \mathbb{Z}^2$
- $\mathbf{s} = (e_1, e_2)$
- $\{e_1, e_2\} = 1$



Scattering Diagrams and Broken Lines

Wall-crossing

The scattering function f is of the form $1 + \sum_k c_k z^{k\{n,\,\cdot\,\}}$ and defines a wall-crossing map $\mathfrak{p}_f : z^m \mapsto z^m f^{\pm\langle m,n\rangle}$, with sign determined by crossing direction.

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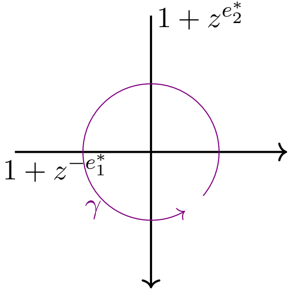

$$z^{e_1^*} \mapsto z^{e_1^*}\left(1 + z^{e_2^* - e_1^*}\right)$$

Not consistent

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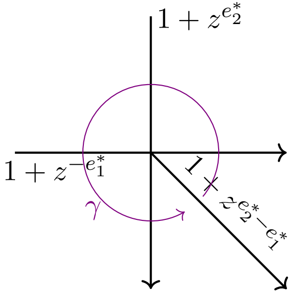


$z^m \mapsto z^m$

Consistent

Scattering Diagrams and Broken Lines

Theorem (Gross-Hacking-Keel-Kontsevich)

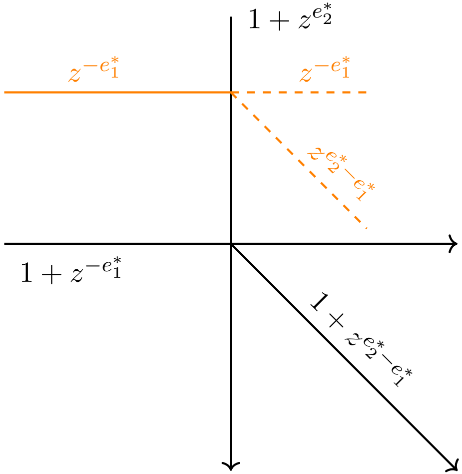
There is a unique (up to equivalence) scattering diagram associated to every cluster variety.

Scattering Diagrams and Broken Lines

Definition

A **broken line** is a piecewise linear map $\gamma : (-\infty, 0] : U^{\mathrm{trop}}(\mathbb{R})$ bending only at walls and having finitely many domains of linearity, which is decorated with a Laurent monomial $cz^{-\gamma'(t)}$ along each linear segment. The coefficient for unbounded segment is 1, and all other decorations are monomial summands arising from wall-crossing.

Scattering Diagrams and Broken Lines



ϑ-functions

From broken lines to ϑ-functions

- Get "ϑ-function" on $U^\vee$ for each $p \in U^{\mathrm{trop}}(\mathbb{Z})$– think N is a basis for $\mathcal{O}(T_M = T_N^\vee)$.

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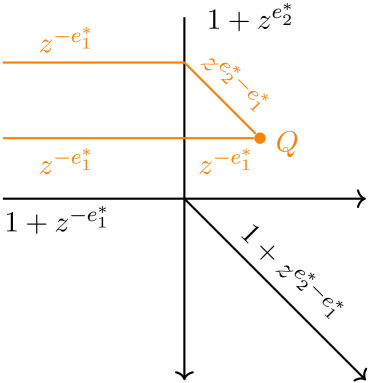
From broken lines to ϑ-functions

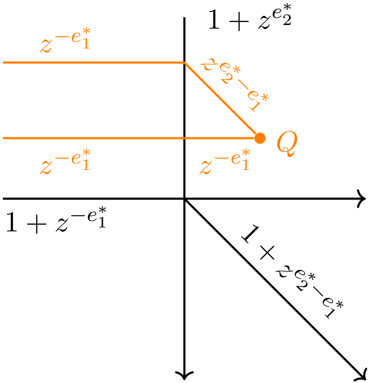
- Get "ϑ-function" on $U^\vee$ for each $p \in U^{\mathrm{trop}}(\mathbb{Z})$– think N is a basis for $\mathcal{O}(T_M = T_N^\vee)$.
- Local coordinates for ϑ_p: pick $Q \in U^{\mathrm{trop}}(\mathbb{R})$. Write ϑ_p as sum of decorating monomials of broken lines starting from direction p and ending at Q.

ϑ-functions



ϑ-functions

$$\vartheta_{-e_1^*} = z^{e_2^* - e_1^*} + z^{-e_1^*}$$



ϑ-function multiplication

Structure constants α_{pq}^r

$$\vartheta_p \cdot \vartheta_q = \sum_{r \in U^{\mathrm{trop}}(\mathbb{Z})} \alpha_{pq}^r \vartheta_r$$

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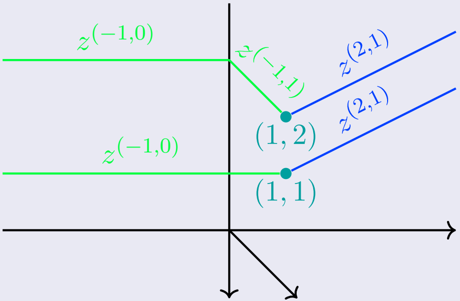
Theorem (Gross-Hacking-Keel-Kontsevich)

$$\alpha_{pq}^r = \sum_{\substack{(\gamma_1,\gamma_2) \\ I(\gamma_1)=p,\ I(\gamma_2)=q \\ \gamma_1(0)=\gamma_2(0)=r \\ F(\gamma_1)+F(\gamma_2)=r}} c(\gamma_1)\ c(\gamma_2)$$

ϑ-function multiplication

Example



$$\vartheta_{(-1,0)} \cdot \vartheta_{(2,1)} = \vartheta_{(1,1)} + \vartheta_{(1,2)}$$

Returning to the conjecture

Conjecture (Gross-Hacking-Keel)

Let U be an affine log Calabi-Yau with maximal boundary. Then

1. we have an algebra A with basis $U^{\mathrm{trop}}(\mathbb{Z})$, where multiplication is given by broken line counts, and

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Let U be an affine log Calabi-Yau with maximal boundary. Then

1. we have an algebra A with basis $U^{\mathrm{trop}}(\mathbb{Z})$, where multiplication is given by broken line counts, and
2. $\mathrm{Spec}(A) = U^{\vee}$.

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What's known?

- If U is a cluster variety, A is constructed in [GHKK18].

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- If U is a cluster variety, A is constructed in [GHKK18]. It is a basis for regular functions on the "Fock-Goncharov dual" cluster variety, which is expected to be the mirror.
- If U contains an open torus, A is constructed [KY19], and they prove "the Frobenius structure conjecture" in this setting.

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What's known?

- If U is a cluster variety, A is constructed in [GHKK18]. It is a basis for regular functions on the "Fock-Goncharov dual" cluster variety, which is expected to be the mirror.
- If U contains an open torus, A is constructed [KY19], and they prove "the Frobenius structure conjecture" in this setting.
- Similar results are proved for the blowup of a toric variety along hypersurfaces in the toric boundary, combining [GS19] and [AG20].

References

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